ESD-TDR-64-564

Technical Report 366

Digital Communication over Fixed Time-Continuous Channels with Memory – with Special Application to Telephone Channels

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20 October 1964

Prepared for the Advanced Research Projects Agency
under Electronic Systems Division Contract AF 19(628)-500 by

Lincoln Laboratory
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Lexington, Massachusetts

The work reported in this document was performed at Lincoln Laboratory, a center for research operated by Massachusetts Institute of Technology. This research is a part of Project Vela Uniform, which is sponsored by the U.S. Advanced Research Projects Agency of the Department of Defense; it is supported by ARPA under Air Force Contract AF 19(628)-500 (ARPA Order 512).

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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Group 64

LINCOLN LABORATORY

TECHNICAL REPORT 366

RESEARCH LABORATORY OF ELECTRONICS

TECHNICAL REPORT 430

20 OCTOBER 1964

LEXINGTON MASSACHUSETTS

DIGITAL COMMUNICATION
OVER FIXED TIME-CONTINUOUS CHANNELS WITH MEMORY –
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ABSTRACT

The objective of this study is to determine the performance, or a bound on the performance, of the "best possible" method for digital communication over fixed time-continuous channels with memory, i.e., channels with intersymbol interference and/or colored noise. The channel model assumed is a linear, time-invariant filter followed by additive, colored Gaussian noise. A general problem formulation is introduced which involves use of this channel once for T seconds to communicate one of M signals. Two questions are considered: (1) given a set of signals, what is the probability of error? and (2) how should these signals be selected to minimize the probability of error? It is shown that answers to these questions are possible when a suitable vector space representation is used, and the basis functions required for this representation are presented. Using this representation and the random coding technique, a bound on the probability of error for a random ensemble of signals is determined and the structure of the ensemble of signals yielding a minimum error bound is derived. The inter-relation of coding and modulation in this analysis is discussed and it is concluded that: (1) the optimum ensemble of signals involves an impractical modulation technique, and (2) the error bound for the optimum ensemble of signals provides a "best possible" result against which more practical modulation techniques may be compared. Subsequently, several suboptimum modulation techniques are considered, and one is selected as practical for telephone channels. A theoretical analysis indicates that this modulation system should achieve a data rate of about 13,000 bits/second on a data grade telephone line with an error probability of approximately 10^{-5}. An experimental program substantiates that this potential improvement could be realized in practice.

Accepted for the Air Force
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* This report is based on a thesis of the same title submitted to the Department of Electrical Engineering at the Massachusetts Institute of Technology in October 1964, in partial fulfillment of the requirements for the Degree of Doctor of Philosophy.

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CHAPTER I

DIGITAL COMMUNICATION OVER TELEPHONE LINES

A. HISTORY

An interest in low-speed digital communication over telephone circuits has existed for many years. As early as 1919, the transmission of teletype and telegraph data had been attempted over both long-distance land lines[1] and transoceanic cables.[2] During these experiments it was recognized that data rates would be severely limited by signal distortion arising from nonlinear phase characteristics of the telephone line. This effect, although present in voice communication, had not been previously noticed due to the insensitivity of the human ear to phase distortion. Recognition of this problem led to fundamental studies by Carson,[3,4] Nyquist,[5,6] and others.[7] From these studies came techniques, presented around 1930, for quantitatively measuring phase distortion[8] and for equalizing lines with such distortion.[9] This work apparently resolved the existing problems, and little or no additional work appears to have been done until the early 1950's.

B. CURRENT INTEREST

The advent of the digital computer in the early 1950's and the resulting military and commercial interest in large-scale information processing systems led to a new interest in using telephone lines for transmitting digital information. This time, however, the high operating speeds of these systems, coupled with the possibility of a widespread use of telephone lines, made it desirable to attempt a more efficient utilization of the telephone channel. Starting about 1954, people at both Lincoln Laboratory[10,11] and Bell Telephone Laboratories[12-14] began investigating this problem. These and other studies were continued by a moderate but ever increasing number of people during the late 1950's.[15-21] By 1960, numerous systems for obtaining high data rates (over 500 bits/second) had been proposed, built, and tested.[22-30] These systems were, however, still quite poor in comparison to what many people felt to be possible. Because of this, and due also to a growing interest in the application of coding techniques to this problem, work has continued at a rapidly increasing pace up to the present.[31-43] Today it is necessary only to read magazines such as <u>Fortune</u>, <u>Business Week</u>, or <u>U. S. News and World Report</u> to observe the widespread interest in this use of the telephone network.[44-47]

C. REVIEW OF CURRENT TECHNOLOGY

The following paragraphs discuss some of the current data transmission systems, or modems (modulator-demodulator), and indicate the basic techniques used along with the resulting performance. Such state-of-the-art information is useful in evaluating the theoretical results obtained in the subsequent analysis.

Two numbers often used in comparing digital communication systems are rate R in bits per second and the probability of error P_e. However, the peculiar properties of telephone channels (Sec. I-D) cause the situation to be quite different here. In fact, most present modems operating on a telephone line whose phase distortion is within "specified limits" have a P_e of 10^{-4} to 10^{-6} that is essentially independent of rate as long as the rate is below some maximum value. As a result, numbers useful for comparison purposes are the maximum rate and the "specified limit" on phase distortion; the latter number is usually defined as the maximum allowable differential delay† over the telephone line passband.

Probably the best-known current modems are those used by the Bell Telephone System in the Data-Phone service.[33] At present, at least two basic systems cover the range from 500 to approximately 2400 bits/second. The simplest of these modems is an FSK system operating at 600 or 1200 bits/second. Alexander, Gryb, and Nast[33] have found that the 600-bits/second system will operate without phase compensation over essentially any telephone circuit in the country and that the 1200-bits/second system will operate over most circuits with a "universal" phase compensator. A second system used by Bell for rates of about 2400 bits/second is a single frequency four-phase differentially modulated system.[48] At present, little additional information is available concerning the sensitivity of this system to phase distortion.

Another system operating at rates of 2400 to 4800 bits/second has been developed by Rixon Electronics, Inc.[24,29] This modem uses binary AM with vestigial side-band transmission and requires telephone lines having maximum differential delays of 200 to 400 μsec.

A third system, the Collins Radio Company Kineplex,[27,49] has been designed to obtain data rates of 4000 to 5000 bits/second. This modem was one of the first to use signal design techniques in an attempt to overcome some of the special problems encountered on the telephone channel. Basically, this system uses four-phase differential modulation of several sinusoidal carriers spaced in frequency throughout the telephone line passband. The differential delay requirements for this system are essentially the same as for the Rixon system at high rate, i.e., about 200 μsec.

Probably the most sophisticated of the modems constructed to date was used at Lincoln Laboratory in a recently reported experiment.[43] In this system, the transmitted signal was "matched" to the telephone line so that the effect of phase distortion was essentially eliminated.‡ Use of this modem with the SECO[50] machine (a sequential coder-decoder) and a feedback channel allowed virtually error-free transmission at an average rate of 7500 bits/second.

D. CHARACTERISTICS OF TELEPHONE LINE AS CHANNEL FOR DIGITAL COMMUNICATION

Since telephone lines have been designed primarily for voice communication, and since the properties required for voice transmission differ greatly from those required for digital transmission, numerous studies have been made to evaluate the properties that are most significant for this application (see Refs. 10, 12, 14, 17, 31, 33). One of the first properties to be recognized was the wide variation of detailed characteristics of different lines. However, later studies

† Absolute time delay is defined to be the derivative, with respect to radian frequency, of the telephone line phase characteristic. Differential delay is defined in terms of this by subtracting out any constant delay. The differential delay for a "typical" telephone line might be 4 to 6 msec at the band edges.

‡ This same approach appears to have been developed independently at IBM.[42]

have shown[33] that only a few phenomena are responsible for the characteristics that affect digital communication most significantly. In an order more or less indicative of their relative importance, these are as follows.

Intersymbol Interference†:— Intersymbol interference is a term commonly applied to an undesired overlap (in time) of received signals that were transmitted as separate pulses. This effect is caused by both the finite bandwidth and the nonlinear phase characteristic of the telephone line, and leads to significant errors even in the absence of noise or to a significant reduction in the signaling rate.‡ It is possible to show, however, that the nonlinear phase characteristic is the primary source of intersymbol interference.

The severity of the intersymbol interference problem can be appreciated from the fact that the maximum rate of current modems is essentially determined by two factors: (1) the sensitivity of the particular signaling scheme to intersymbol interference, and (2) the "specified limit" on phase distortion; the latter being in some sense a specification of allowable intersymbol interference. In none of these systems does noise play a significant role in determining rate as it does, for example, in the classical additive, white Gaussian noise channel.[51] Thus, current modems trade rate for sensitivity to phase distortion — a higher rate requiring a lower "specified limit" on phase distortion and vice versa.

Impulse and Low-Level Noise:— Experience has shown that the noise at the output of a telephone line appears to be the sum of two basic types.[31,33] One type of noise, low-level noise, is typically 20 to 50 db below normal signal levels and has the appearance of Gaussian noise superimposed on harmonics of 60 cps. The level and character of this noise is such that it has negligible effect on the performance of current modems. The second type of noise, impulse noise, differs from low-level noise in several basic attributes.[52] First, its appearance when viewed on an oscilloscope is that of rather widely separated (on the order of seconds, minutes, or even days) bursts of relatively long (on the order of 5 to 50 msec) transient pulses. Second, the level of impulse noise may be as much as 10 db above normal signal levels. Third, impulse noise appears difficult to characterize in a statistical manner suitable for deriving optimum detectors. Because of these characteristics, present modems make little or no attempt to combat impulse noise; furthermore, impulse noise is the major source of errors in these systems. In fact, most systems operating at a rate such that intersymbol interference is a negligible factor in determining probability of error will be found to have an error rate almost entirely dependent upon impulse noise — typical error rates being 1 in 10^4 to 1 in 10^6 (Ref. 33).

Phase Crawl:— Phase crawl is a term applying to the situation in which the received signal spectrum is displaced in frequency from the transmitted spectrum. Typical displacements are from 0 to 10 cps and arise from the use of nonsynchronous oscillators in frequency translations performed by the telephone company. Current systems overcome this effect by various modulation techniques such as AM vestigial sideband, differentially modulated FM, and carrier recovery with re-insertion.

Dropout:— The phenomena called dropout occurs when for some reason the telephone line appears as a noisy open circuit. Dropouts are usually thought to last for only a small fraction of a

† Implicit in the following discussion of intersymbol interference is the assumption that the telephone line is a linear device. Although this may not be strictly true, it appears to be a valid approximation in most situations.

‡ Alternately, and equivalently, intersymbol interference can be viewed in the time domain as arising from the long impulse response of the line (typically 10 to 15 msec duration).

second, although an accidental opening of the line can clearly lead to a much longer dropout. Little can be done to combat this effect except for the use of coding techniques.

Crosstalk:— Crosstalk arises from electromagnetic coupling between two or more lines in the same cable. Currently, this is a secondary problem relative to intersymbol interference and impulse noise.

The previous discussion has indicated the characteristics of telephone lines that affect digital communication most significantly. It must be emphasized, however, that present modems are limited in performance almost entirely by intersymbol interference and impulse noise. The maximum rate is determined primarily by intersymbol interference and the probability of error is determined primarily by impulse noise. Thus, an improved signaling scheme that considerably reduces intersymbol interference should allow a significant increase in data rate with a negligible increase in probability of error. Some justification for believing that this improvement is possible in practice is given by the experiment at Lincoln Laboratory.[43] In this experiment, a combination of coding and a signal design that reduced intersymbol interference allowed performance significantly greater than any achieved previously. Even so, the procedure for combating intersymbol interference was ad hoc. Thus, the primary objective of this report is to obtain a fundamental theoretical understanding of optimum signaling techniques for channels whose characteristics are similar to those of the telephone line.

E. MATHEMATICAL MODEL FOR DIGITAL COMMUNICATION OVER FIXED TIME-CONTINUOUS CHANNELS WITH MEMORY

1. Introduction

Basic to a meaningful theoretical study of a real life problem is a model that includes the important features of the real problem and yet is mathematically tractable. This section presents a relatively simple, but heretofore incompletely analyzed, model that forms the basis for the subsequent theoretical work. There are two fundamental reasons for this choice of model:

(a) It represents a generalization of the classical white, Gaussian noise channel considered by Fano,[51] Shannon,[53] and others. Thus, any analysis of this channel represents a generalization of previous work and is of interest independently of any telephone line considerations.

(b) As indicated previously, the performance of present telephone line communication systems is limited in rate by intersymbol interference and in probability of error by impulse noise; the low-level "Gaussian" noise has virtually no effect on system performance. However, the frequent occurrence of long intervals without significant impulse noise activity makes it desirable to study a channel which involves only intersymbol interference (it is time dispersive) and Gaussian noise. In this manner, it will be possible to learn how to reduce intersymbol interference and thus increase rate to the point where errors caused by low-level noise are approximately equal in number to errors caused by impulse noise.

2. Some Considerations in Choosing a Model

One of the fundamental aims of the present theoretical work is to determine the performance, or a bound on the performance, of the "best possible" method for digital communication over fixed time-continuous channels with memory, i.e., channels with intersymbol interference

and/or colored noise. In keeping with this goal, it is desirable to include in the model only those features that are fundamental to the problem when all practical considerations are removed. For example, practical constraints often require that digital communication be accomplished by the serial transmission of short, relatively simple pulses having only two possible amplitudes. The theoretical analysis will show, however, that for many channels this leads to an extremely inefficient use of the available channel capacity. In other situations, when communication over a narrow-band, bandpass channel is desired, it is often convenient to derive the transmitted signal by using a baseband signal to amplitude, phase or frequency modulate a sinusoidal carrier. However, on a wide-band, bandpass channel such as the telephone line it is not a priori clear that this approach is still useful or appropriate although it is certainly still possible. Finally, it should be recognized that the interest in a model for digital† communication implies that detection and decision theory concepts are appropriate as opposed to least-mean-square error filtering concepts that find application in analog communication.

3. Model

An appropriate model for digital communication over fixed time-dispersive channels can be specified in the following manner. An obvious but fundamental fact is that in any real situation it is necessary to transmit information for only a finite time, say T seconds. This, coupled with the fact that a model for digital communication is desired, implies that one of only a finite number, say M, of possible messages is to be transmitted.‡ For the physical situation being considered, it is useful to think of transmitting this message by establishing a one-to-one correspondence between a set of M signals of T seconds duration and transmitting the signal that corresponds to the desired message. Furthermore, in any physical situation, there is only a finite amount of energy, say ST, available with which to transmit the signal. (Implicit here is the interpretation of S as average signal power.) This fact leads to the assumption of some form of an energy constraint on the set of signals – a particularly convenient constraint being that the statistical average of the signal energies is no greater than ST. Thus, if the signals are denoted by $s_i(t)$ and each signal is transmitted with probability P_i, the constraint is

$$\sum_{i=1}^{M} P_i \int_0^T s_i^2(t)\, dt \leq ST \quad . \tag{1}$$

Next, the time-dispersive nature of the channel must be included in the model. A model for this effect is simply a linear time-invariant filter. The only assumption required on this filter is that its impulse response have finite energy, i.e., that

$$\int_{-\infty}^{\infty} h^2(t)\, dt < \infty \quad . \tag{2}$$

† The word "digital" is used here and throughout this work to mean that there are only a finite number of possible messages in a finite time interval. It should not be construed to mean "the transmission of binary symbols" or anything equally restrictive.

‡ At this point, no practical restrictions will be placed on T or M. So, for example, perfectly allowable values for T and M might be $T = 3 \times 10^7$ seconds $\simeq$ 1 year and $M = 10^{11}$. This is done to allow for a very general formulation of the problem. Later analysis will consider more practical situations.

It is convenient, however, to assume, as is done through this work, that the filter is normalized so that $\max_f |H(f)| = 1$, where

$$H(f) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt \quad .$$

Furthermore, to make the entire problem nontrivial, some noise must be considered. Since the assumption of Gaussian noise leads to mathematically tractable results and since a portion of the noise on telephone lines appears to be "approximately" Gaussian, this form for the noise is assumed in the model. Moreover, since actual noise appears to be additive, that is also assumed. For purposes of generality, however, the noise will be assumed to have an arbitrary spectral density $N(f)$.

Finally, to enable the receiver to determine the transmitted message it is necessary to observe the received signal (the filtered transmitted signal corrupted by the additive noise) over an interval of, say T_1 seconds, and to make a decision based upon this observation.†

In summary, the model to be analyzed is the following: given a set of M signals of T seconds duration satisfying the energy constraint of Eq. (1), a message is to be transmitted by selecting one of the signals and transmitting it through the linear filter $h(t)$. The filter output is assumed to be corrupted by (possibly colored) Gaussian noise and the receiver is to decide which message was transmitted by observing the corrupted signal for T_1 seconds.

Given the above model, a meaningful performance criterion is probability of error. On the basis of this criterion, three fundamental questions can be posed.

> Given a set of signals $\{s_i(t)\}$, what form of decision device should be used?
>
> What is the resulting probability of error?
>
> How should a set of signals be selected to minimize the probability of error?

The answer to the first question involves well-known techniques[54] and will be discussed only briefly in Chapter III. The determination of the answers to the remaining two questions is the primary concern of the theoretical portion of this report.

In conclusion, it must be emphasized that the problem formulated in this section is quite general. Thus, it allows for the possibility that optimum signals may be of the form of those used in current modems. The formulation has not, however, included any practical constraints on signaling schemes and thus does not preclude the possibility that an alternate and superior technique may be found.

† At this point, T_1 is completely arbitrary. Later it will prove convenient to assume $T_1 \geqslant T$ which is the situation of most practical interest.

CHAPTER II
SIGNAL REPRESENTATION PROBLEM

A. INTRODUCTION

The previous section presented a model for digital communication over fixed time-dispersive channels and posed three fundamental questions concerning this model However, an attempt to obtain detailed answers to these questions involves considerable difficulty. The source of this difficulty is that the energy constraint is applied to signals at the filter input, whereas the probability of error is determined by the structure of these signals at the filter output. Fortunately, the choice of a signal representation that is "matched" to both the model and the desired analysis allows the presence of the filter to be handled in a straightforward manner. The following sections present a brief discussion of the general signal representation problem, slanted, of course, toward the present analysis, and provide the necessary background for subsequent work.

B. SIGNAL REPRESENTATION

At the outset, it should be mentioned that many of the concepts, techniques, and terminology of this section are well known to mathematicians under the title of "Linear Algebra."[55]

As pointed out by Siebert,[56] the fundamental goal in choosing a signal representation for a given problem is the simplification of the resulting analysis. Thus, for a digital communication problem, the signal representation is chosen primarily to simplify the evaluation of probability of error. One representation which has been found to be extremely useful in such problems (due largely to the widespread assumption of Gaussian noise) pictures signals as points in an n-dimensional Euclidean vector space.

1. Vector Space Concept

In a digital communication problem it is necessary to represent a finite number of signals M. One way to accomplish this is to write each signal as a linear combination of a (possibly infinite) set of orthonormal "basis" functions $\{\varphi_i(t)\}$, i.e.,

$$s_j(t) = \sum_{i=1}^{n} s_{ij}\varphi_i(t) \tag{3}$$

where

$$s_j = \int \varphi_i(t)\, s_j(t)\, dt \quad .$$

When this is done, it is found in many cases that the resulting probability of error analysis depends <u>only</u> on the numbers s_{ij} and is independent of the basis functions $\{\varphi_i(t)\}$. In such cases, a vector or n-tuple $\underline{s}_j$ can be defined as

$$\underline{s}_j = (s_{1j}, s_{2j}, \ldots, s_{kj}, \ldots, s_{nj})$$

which, in so far as the analysis is concerned, represents the time function $s_j(t)$. Thus, it is possible to view $\underline{s}_j$ as a straightforward generalization of a three-dimensional vector and $s_j(t)$ as a vector in an n-dimensional vector space. The utility of this viewpoint is clear from its

widespread use in the literature. Two basic reasons for this usefulness, at least in problems with Gaussian noise, are clear from the following relations which are readily derived from Eq. (3). The energy of a signal $s_j(t)$ is given by

$$\int s_j^2(t)\, dt = \sum_{i=1}^{n} s_{ij}^2 = \underline{s}_j \cdot \underline{s}_j$$

and the cross correlation between two signals $s_i(t)$ and $s_j(t)$ is given by

$$\int s_j(t)\, s_k(t)\, dt = \sum_{i=1}^{n} s_{ij} s_{ik} = \underline{s}_j \cdot \underline{s}_k$$

where ( ) · ( ) denotes the standard inner product.[55]

2. Choice of Basis Functions

So far, the discussion of the vector space representation has been concerned with basic results from the theory of orthonormal expansions.[57] However, an attempt to answer a related question – how are the $\varphi_i(t)$ to be chosen – leads to results that are far less well defined and less well known. Fundamentally, this difference arises because the choice of the $\varphi_i(t)$ depends heavily upon the type of analysis to be performed, i.e., the $\varphi_i(t)$ should be chosen to "simplify the analysis as much as possible." Since such a criterion clearly leads to no specific rule for determining the $\varphi_i(t)$, it is possible only to indicate situations in which distinctly different basis functions might be appropriate.

Band-Limited Signals:– A set of basis functions used widely for representing band-limited signals is the set of $\varphi_i(t)$ defined by

$$\varphi_i(t) = \sqrt{2W}\, \frac{\sin 2\pi W\, [t - (i/2W)]}{2\pi W\, [t - (i/2W)]} \tag{4}$$

where W is the signal bandwidth. The popularity of this representation, the so-called sampling representation, lies almost entirely in the simple form for the coefficient s_{ij}. This is readily shown to be

$$s_{ij} = \int_{-\infty}^{\infty} \varphi_i(t)\, s_j(t)\, dt = \frac{1}{\sqrt{2W}}\, s_j(i/2W) \quad . \tag{5}$$

It should be recalled, however, that no physical signal can be precisely band-limited.[58] Thus, any attempt to represent a physical signal by this set of basis functions must give only an approximate representation. However, it is possible to make the approximation arbitrarily accurate by choosing W sufficiently large.

Time-Limited Signals:– Time-limited signals are often represented by any one of several forms of a Fourier series. These representations are well known to engineers and any discussion here would be superfluous. It is worth noting that this representation, in contrast to the sampling representation, is exact for any signal of engineering interest.

Arbitrary Set of M Signals:– The previously described representations share the property that, in general, an infinite number of basis functions are required to represent a finite number of signals. However, in problems involving only a finite number of signals, it

is sometimes convenient to choose a different set of basis functions so that no more than M basis functions are required to represent M signals. A proof that such a set exists, along with the procedure for finding the functions, has been presented by Arthurs and Dym.[59] This result, although well known to mathematicians, appears to have just recently been recognized by electrical engineers.

Signals Corrupted by Additive Colored Gaussian Noise:— In problems involving signals of T seconds duration imbedded in colored Gaussian noise, it is often desirable to represent both signals and noise by a set of basis functions such that the noise coefficients are statistically independent random variables. If the noise autocorrelation function† is $R(\tau)$, it is well known from the Karhunen-Loeve theorem[54,60] that the $\varphi_i(t)$ satisfying the integral equation

$$\lambda_i \varphi_i(\tau) = \int_0^T \varphi_i(\tau') \, R(\tau - \tau') \, d\tau \qquad 0 \leqslant \tau \leqslant T$$

form such a set of basis functions.

Filtered Signals:— Consider the following somewhat artificial situation closely related to the results of Sec. II-D. A set of M finite energy signals defined on the interval $[0, T]$ is given. Also given is a nonrealizable linear filter whose impulse response satisfies $h(t) = h(-t)$. It is desired to represent both the given signals and the signals that are obtained when these are passed through the filter by orthonormal expansions defined on the interval $[0, T]$. In general, arbitrary and different sets of $\varphi_i(t)$ can be chosen for both representations. Then the relation between the input signal vector $\underline{s}_j$ and the output signal vector, say $\underline{r}_j$, is determined as follows:

Let $r_j(t)$ be the filter output when $s_j(t)$ is the filter input. Then

$$r_j(t) \triangleq \int_0^T s_j(\tau) \, h(t - \tau) \, d\tau = \sum_i s_{ij} \int_0^T \alpha_i(\tau) \, h(t - \tau) \, d\tau$$

where $\{\alpha_i(t)\}$ are the basis functions for the input signals. Thus, if $\{\gamma_i(t)\}$ are the basis functions for the output signals, it follows that

$$r_{kj} \triangleq \int_0^T r_j(t) \, \gamma_k(t) \, dt = \sum_i s_{ij} \int_0^T \int_0^T \gamma_k(t) \, h(t - \tau) \, \alpha_i(\tau) \, d\tau \, dt$$

or, in vector notation,

$$\underline{r}_j = [H] \, \underline{s}_j \tag{6}$$

where the k, i^{th} element in $[H]$ is

$$\int_0^T \int_0^T \gamma_k(t) \, h(t - \tau) \, \alpha_i(\tau) \, d\tau \, dt$$

and, in general, $\underline{r}_j$ and $\underline{s}_j$ are infinite dimensional column vectors and $[H]$ is an infinite dimensional matrix.

† For the statement made here to be strictly true, it is sufficient that $R(\tau)$ be the autocorrelation function of filtered white noise.[61]

If, however, a common set of basis functions is used for both input and output signals and if these $\varphi_i(t)$ are taken to be the solutions of the integral equation†

$$\lambda_i \varphi_i(t) = \int_0^T \varphi_i(\tau)\, h(t-\tau)\, d\tau \qquad 0 \leqslant t \leqslant T$$

then the relation of Eq. (6) will still be true but now [H] will be a diagonal matrix with the eigenvalues λ_i along the main diagonal. This result, which is related to the spectral decomposition of linear self-adjoint operators,[62,63] has two important features. First, and most obvious, the diagonalization of the matrix [H] leads to a much simplified calculation of $\underline{r}_j$ given $\underline{s}_j$. Equally important, however, this form for [H] has entries depending only upon the filter impulse response h(t). Although not a priori obvious, these two features are precisely those required of a signal representation to allow evaluation of probability of error for digital transmission over time-dispersive channels.

C. DIMENSIONALITY OF FINITE SET OF SIGNALS

This section concludes the general discussion of the signal representation problem by presenting a definition of the dimensionality of a finite set of signals that is of independent interest and is, in addition, of considerable use in defining the dimensionality of a communication channel.

An approximation often used by electrical engineers is given by the statement that a signal which is "approximately" time-limited to T seconds and "approximately" band-limited to W cps has 2TW "degrees of freedom"; i.e., that such a signal has a "dimensionality" of 2TW. This approximation is usually justified by a conceptually simple but mathematically unappealing argument based upon either the sampling representation or the Fourier series representation previously discussed. However, fundamental criticisms of this statement make it desirable to adopt a different and mathematically precise definition of "dimensionality" that overcomes these criticisms and yet retains the intuitive appeal of the statement. Specifically, these criticisms are:

(1) If a (strictly) band-limited nonzero signal is assumed, it is known[58] that this signal must be nonzero over any time interval of nonzero length. Thus, any definition of the "duration" T of such a signal must be arbitrary, implying an arbitrary "dimensionality," or equally unappealing, the signal must be considered to be of infinite duration and therefore of infinite "dimensionality." Conversely, if a time-limited signal is assumed, it is known[58] that its energy spectrum exists for all frequencies. Thus, any attempt to define "bandwidth" for such a signal leads to similar problems. Clearly, the situation in which a signal is neither band-limited nor time-limited, e.g., $s(t) = \exp[-|t|]$ where $-\infty < t < \infty$, leads to even more difficulties.‡

(2) The fundamental importance of the concept of the "dimensionality" of a signal is that it indicates, hopefully, how many real numbers must be given to specify the signal. Thus, when signals are represented as points in n-dimensional space, it is often useful to define the dimensionality of a signal to be the dimensionality of the corresponding vector space. This

† Again, there are mathematical restrictions on h(t) before the following statements are strictly true. These conditions[64,65] are concerned with the existence and completeness of the $\phi_i(t)$ and are of secondary interest at this point.

‡ It should be mentioned that identical problems are encountered when an attempt is made to define the "dimensionality" of a channel in a similar manner. This problem will be discussed in detail in Chapter III.

definition, however, may lead to results quite different from those obtained using the concept of "duration" and "bandwidth." For example, consider an arbitrary finite energy signal s(t). Then by choosing for an orthonormal basis the <u>single</u> function

$$\varphi_1(t) = \frac{s(t)}{\left[\int_{-\infty}^{\infty} s^2(t)\, dt\right]^{1/2}}$$

it follows that

$$s(t) = s_1 \varphi_1(t)$$

where, as usual,

$$s_1 = \int_{-\infty}^{\infty} s(t)\, \varphi_1(t)\, dt \quad .$$

Thus, this definition of the dimensionality of s(t) indicates that it is only one dimensional in contrast to the arbitrary (or infinite) dimensionality found previously. Clearly, such diversity of results leaves something to be desired.

Although this discussion may seem somewhat confusing and puzzling, the reason for the widely different results is readily explained. Fundamentally, the time-bandwidth definition of signal dimensionality is an attempt to define the "useful" dimensionality of the vector space obtained when the basis functions are restricted to be either the band-limited $\sin x/x$ functions or the time-limited sine and cosine functions. In contrast, the second definition of dimensionality allowed an arbitrary set of basis functions and in doing so allowed the $\varphi_i(t)$ to be chosen to minimize the dimensionality of the resulting vector space.

In view of the above discussions, and because it will prove useful later, the following definition for the dimensionality of a set of signals will be adopted:†

Let S be a set of M finite energy signals and let each signal in this set be represented by a linear combination of a set of orthonormal functions, i.e.,

$$s_j(t) = \sum_{i=1}^{N} s_{ij} \varphi_i(t) \qquad \text{for all } s_j(t) \in S \quad .$$

Then the <u>dimensionality</u> d of this set of signals is defined to be the minimum of N over all possible basis functions, i.e.,

$$d \triangleq \min_{\{\varphi_i(t)\}} N \quad .$$

The proof that such a number d exists, that $d \leq M$, and the procedure for finding the $\{\varphi_i(t)\}$ have been presented elsewhere[59] and will not be considered here. It should be noted, however, that the definition given is unambiguous and, as indicated, is quite useful in the later work. It is also satisfying to note that if S is a <u>set</u> of band-limited signals having the property that

$$s_j(i/2W) = 0 \qquad \begin{cases} \text{for all } i < 1 \text{ or } i > 2TW \\ \text{for all } s_j(t) \in S \end{cases}$$

†This definition is just the translation into engineering terminology of a standard definition of linear algebra.[55,66]

then the above definition of dimensionality leads to d = 2TW. A similar result is also obtained for a set of time-limited functions whose frequency samples all vanish except for a set of 2TW values.

D. SIGNAL REPRESENTATION FOR FIXED TIME-CONTINUOUS CHANNELS WITH MEMORY

In Sec. I-E, it was demonstrated that a useful model for digital communication over fixed time-continuous channels with memory considers the transmission of signals of T-seconds duration through the channel of Fig. 1 and the observation of the received signal y(t) for T_1 seconds. Given this model, the problem is to determine the probability of error for an optimum detector and a particular set of signals and then to minimize the probability of error over the available sets of signals. Under the assumption that the vector space representation is appropriate for this situation, there remains the problem of selecting the basis functions for both the transmitted signals x(t) and the received signals y(t).

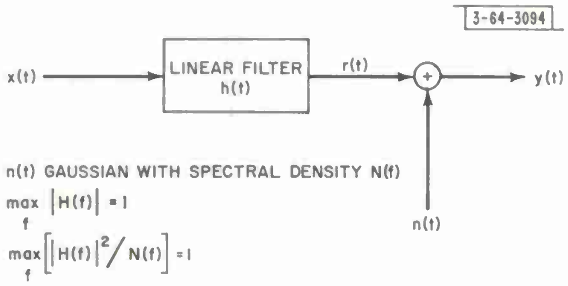

Fig. 1. Time-continuous channel with memory.

If $\underline{x}$, $\underline{n}$, and $\underline{y}$ are the (column) vector representations of the transmitted signal (assumed to be defined on 0, T), additive noise, and received signal (over the observation interval of T_1 seconds), respectively, an arbitrary choice of basis functions will lead to the vector equation

$$\underline{y} = [H]\,\underline{x} + \underline{n} \tag{7}$$

in which all vectors are, in general, infinite dimensional, $\underline{n}$ may have correlated components, and [H] will be an infinite dimensional matrix related to the filter impulse response and the sets of basis functions selected.† If, instead, both sets of basis functions are selected in the manner presented here and elsewhere,[67] it will be found (1) that [H] is a diagonal matrix whose entries are the square root of the eigenvalues of a related integral equation, and (2) that the components of $\underline{n}$ are statistically independent and identically distributed Gaussian random variables. Because of these two properties it is possible to obtain a meaningful and relatively simple bound on probability of error for the channel considered in this work.

In the following discussion it would be possible, at least in principle, to present a single procedure for obtaining the desired basis functions which would be valid for any filter impulse response, any noise spectral density, and any observation interval, finite or infinite. This approach, however, leads to a number of mathematically involved limiting arguments when white noise and/or an infinite observation interval is of interest. Because of these difficulties the following situations are considered separately and in the order indicated.

†It is, of course, possible to obtain statistically independent noise components by using for the receiver signal space basis functions the orthonormal functions used in the Karhunen-Loeve expansion of the noise.[54,60] However, this will not, in general, diagonalize [H].

Arbitrary filter, white noise, arbitrary observation interval (arbitrary T_1);
Arbitrary filter, colored noise, infinite observation interval ($T_1 = \infty$);
Arbitrary filter, colored noise, finite observation interval ($T_1 < \infty$).

Due to their mathematical nature, the main results of this section are presented in the form of several theorems. First, however, some assumptions and simplifying notation will be introduced.

Assumptions.

(1) The time functions h(t), x(t), y(t), and n(t) are in all cases real.

(2) The input signal x(t) may be nonzero only on the interval [0, T] and has finite energy; that is, $x(t) \in \mathcal{L}_2(0,T)$ and thus

$$\int_0^T x^2(t)\, dt = \int_{-\infty}^{\infty} x^2(t)\, dt < \infty \quad .$$

(3) The filter impulse response h(t) is physically realizable and stable.[56] Thus, $h(t) = 0$ for $t < 0$ and

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty \quad .$$

(4) The time scale for y(t) is shifted to remove any pure delay in h(t).

(5) The observation interval for y(t) is the interval $[0, T_1]$, unless otherwise indicated.

Note: Assumptions 3, 4, and 5 have been made primarily to simplify the proof that the $\varphi_i(t)$ are complete. Clearly, these assumptions cause no loss of generality with respect to "real world" communication problems. Furthermore, completeness proofs, although considerably more tedious, are possible only under the assumption that

$$\int_{-\infty}^{\infty} h^2(t)\, dt < \infty \quad .$$

Notation.

The standard inner product on the interval [0, T] is written (f, g); that is,

$$(f, g) \triangleq \int_0^T f(t)\, g(t)\, dt \quad .$$

The linear integral operation on f(t, s) by $k(t, \tau)$ is written kf(t, s); that is,

$$kf(t, s) \triangleq \int_{-\infty}^{\infty} k(t, \tau)\, f(\tau, s)\, d\tau \quad .$$

The generalized inner product on the interval $[0, T_1]$ is written $(f, kg)_{T_1}$; that is,

$$(f, kg)_{T_1} \triangleq \int_0^{T_1} f(t)\, kg(t)\, dt = \int_0^{T_1} f(t) \int_{-\infty}^{\infty} k(t, \tau)\, g(\tau)\, d\tau\, dt \quad .$$

With these preliminaries the pertinent theorems can now be stated. The following results are closely related to the spectral decomposition of linear self-adjoint operators on a Hilbert

space.[62,63,66] It should be mentioned that all the following results can be applied directly to time-discrete channels by simply replacing integral operators with matrix operators.

1. Basis Functions for White Noise and an Arbitrary Observation Interval

Theorem 1.

Let $N(f) = 1$, define a symmetric function $K(t,s) = K(s,t)$ in terms of the filter impulse response by

$$K(t,s) \triangleq \begin{cases} \int_0^{T_1} h(\tau - t)\, h(\tau - s)\, d\tau & 0 \leq t, s \leq T \\ 0 & \text{elsewhere} \end{cases}$$

and define a set of eigenfunctions and eigenvalues by

$$\lambda_i \varphi_i(t) = K\varphi_i(t) \qquad \begin{cases} 0 \leq t \leq T \\ i = 1, 2, \ldots \end{cases} \quad . \tag{8}$$

[Here and throughout the remainder of this work, it is assumed that $\varphi_i(t) = 0$ for $t < 0$ and $t > T$, $i = 1, 2, \ldots$.] Then the vector representation of an arbitrary $x(t) \in \mathfrak{L}_2(0,T)$ is $\underline{x}$, where $x_i = (x, \varphi_i)$ and the vector representation of the corresponding† $y(t)$ on the interval $[0, T_1]$ $(T_1 \geq T)$‡ is given by Eq. (7) in which the components of $\underline{n}$ are statistically independent and identically distributed Gaussian random variables with zero-mean and unit variance and

$$[H] = \begin{bmatrix} \sqrt{\lambda_1} & & & & 0 \\ & \sqrt{\lambda_2} & & & \\ & & \ddots & & \\ & & & \sqrt{\lambda_n} & \\ 0 & & & & \ddots \end{bmatrix}$$

The basis functions for $y(t)$ are $\{\theta_i(t)\}$, where

$$\theta_i(t) = \begin{cases} \dfrac{1}{\sqrt{\lambda_i}}\, h\varphi_i(t) & 0 \leq t \leq T_1 \\ 0 & \text{elsewhere} \end{cases}$$

and the i^{th} component of $\underline{y}$ is

$$y_i = (y, \theta_i)_{T_1} \quad .$$

Note: To be consistent with the literature,[64,65] it is necessary to denote as eigenfunctions only those solutions of Eq. (8) having $\lambda_i > 0$. This restriction is required since mathematicians normally place the eigenvalue on the right-hand side of Eq. (8) and do not consider eigenfunctions with infinite eigenvalue.

---

† The representation for $y(t)$ neglects a noise "remainder term" which is irrelevant in the present work. See the discussion following the proof of Lemma 3 for a detailed consideration of this point.

‡ All the following statements will be true if $T_1 < T$ except that the $\{\phi_i(t)\}$ will not be complete in $\mathfrak{L}_2(0,T)$.

Proof.

The proof of this theorem consists of a series of Lemmas. The first Lemma demonstrates that eigenfunctions of Eq. (8) form a basis for $\mathcal{L}_2(0, T)$, i.e., that they are complete.

Lemma 1(a).

If $T_1 \geqslant T$, the set of functions $\{\varphi_i(t)\}$ defined by Eq. (8) form an orthonormal basis for $\mathcal{L}_2(0, T)$, that is, for every $x(t) \in \mathcal{L}_2(0, T)$

$$x(t) = \sum_{i=1}^{\infty} x_i \varphi_i(t) \qquad 0 \leqslant t \leqslant T$$

where $x_i = (x, \varphi_i)$ and mean-square convergence is obtained.

Proof.

Since the kernel of Eq. (8) is $\mathcal{L}_2$ (see Appendix A) and symmetric, it is well known[64] that at least one eigenfunction of Eq. (8) is nonzero, that all nonzero and nondegenerate eigenfunctions are orthogonal (and therefore may be assumed orthonormal) and that degenerate eigenfunctions are linearly independent and of finite multiplicity and may be orthonormalized. Furthermore, it is known[61,65] that the $\{\varphi_i(t)\}$ are complete in $\mathcal{L}_2(0, T)$ if and only if the condition

$$(f, Kf)_{T_1} = 0 \qquad f(t) \in \mathcal{L}_2(0, T)$$

implies $f(t) = 0$ almost everywhere on $[0, T]$. But

$$(f, Kf)_{T_1} = \int_0^{T_1} \left[\int_0^T f(\tau)\, h(t-\tau)\, d\tau\right]^2 dt \quad .$$

Thus, to prove completeness, it suffices to prove that if

$$\int_0^T f(\tau)\, h(t-\tau)\, d\tau = 0 \qquad 0 \leqslant t \leqslant T_1 \text{ with } T_1 \geqslant T$$

then $f(t) = 0$ almost everywhere on $[0, T]$. Let

$$u(t) = \int_0^T f(\tau)\, h(t-\tau)\, dt$$

and assume that

$$u(t) = \begin{cases} 0 & t \leqslant T_1 \\ z(t - T_1) & t > T_1 \end{cases}$$

where $z(t)$ is zero for $t < 0$ and is arbitrary elsewhere, except that it must be the result of passing some $\mathcal{L}_2(0, T)$ signal through $h(t)$. Then

$$U(s) = \int_0^{\infty} u(t)\, e^{-st}\, dt = e^{-sT_1} Z(s) = F(s)\, H(s) \qquad \text{Re}[s] \geqslant 0 \quad . \tag{9}$$

Now, for $\text{Re}[s] \geqslant 0$,

$$|Z(s)| = \left|\int_0^\infty z(t)\, e^{-st}\, dt\right| \leqslant \int_0^\infty |z(t)| \exp\{-\mathrm{Re}[s]\, t\}\, dt$$

$$\leqslant \int_0^\infty |z(t)|\, dt = \int_0^\infty \left|\int_0^T f(\tau)\, h(t-\tau)\, d\tau\right| dt$$

$$\leqslant \int_0^T |f(\tau)| \int_0^\infty |h(t-\tau)|\, dt\, d\tau \leqslant \int_0^T |f(\tau)|\, d\tau \int_{-\infty}^\infty |h(t)|\, dt \quad .$$

However, from the Schwarz inequality,

$$\int_0^T |f(\tau)|\, d\tau \leqslant \left[\int_0^T d\sigma \int_0^T f^2(\tau)\, d\tau\right]^{1/2} ,$$

and, from assumption 3,

$$\int_{-\infty}^\infty |h(t)|\, dt < \infty \quad .$$

Thus,

$$|Z(s)| \leqslant \left[T \int_0^T f^2(\tau)\, d\tau\right]^{1/2} \int_{-\infty}^\infty |h(t)|\, dt < \infty \qquad \mathrm{Re}[s] \geqslant 0 \quad .$$

Since $f(t) \in \mathcal{L}_2(0, T)$, this result combined with Eq. (9) implies that there exists a constant A such that

$$|F(s)\, H(s)| \leqslant A \exp\{-\mathrm{Re}[s]\, T_1\} \qquad \text{for } \mathrm{Re}[s] \geqslant 0 \quad .$$

From a Lemma of Titchmarsh[68] it follows that there exist constants α and β with $\alpha + \beta = T_1$, such that

$$|F(s)| \leqslant A_1 \exp\{-\mathrm{Re}[s]\, \alpha\} \qquad \mathrm{Re}[s] \geqslant 0$$

$$|H(s)| \leqslant A_2 \exp\{-\mathrm{Re}[s]\, \beta\} \qquad \mathrm{Re}[s] \geqslant 0$$

where $A_1 A_2 = A$. Finally, since

$$f(t) = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} F(s)\, e^{st}\, ds$$

and similarly

$$h(t) = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} H(s)\, e^{st}\, ds \quad ,$$

these conditions and ordinary contour integration around a right half-plane contour imply that

$$f(t) = 0 \qquad t < \alpha$$

$$h(t) = 0 \qquad t < \beta \quad .$$

From assumptions 3 and 4, h(t) is physically realizable and contains no pure delay. Thus, by choosing $\beta = 0$, it follows that $\alpha = T_1$ and, if $T_1 \geqslant T$, that $f(t) = 0$ almost everywhere on $[0, T]$. This completes the proof that the $\{\varphi_i(t)\}$ are complete.

The following Lemma demonstrates that the $\{\Theta_i(t)\}$ defined in the statement of the theorem are a basis for all signals at the filter output, i.e., all signals of the form

$$r(t) = \int_0^T x(\tau)\, h(t-\tau)\, d\tau \qquad \begin{cases} 0 \leqslant t \leqslant T_1 \\ x(t) \in \mathcal{L}_2(0,T) \end{cases} .$$

Lemma 1(b).

Let $r(t)$ and $\Theta_i(t)$ be as defined previously. Then

$$r(t) = \sum_{i=1}^{\infty} r_i \Theta_i(t) \qquad 0 \leqslant t \leqslant T_1$$

where $r_i = (r, \Theta_i)_{T_1} = \sqrt{\lambda_i}\, x_i$, $x_i = (x, \varphi_i)$, and uniform convergence[69] is obtained.

Proof.

Define two functions $r_n(t)$ and $x_n(t)$ by

$$x_n(t) \triangleq \sum_{i=1}^{n} x_i \varphi_i(t) \qquad 0 \leqslant t \leqslant T$$

where

$$x_i = (x, \varphi_i)$$

and

$$r_n(t) \triangleq \int_0^T x_n(\tau)\, h(t-\tau)\, dt \qquad 0 \leqslant t \leqslant T_1 .$$

Then

$$|r(t) - r_n(t)| = \left| \int_0^T [x(\tau) - x_n(\tau)]\, h(t-\tau)\, d\tau \right|$$

$$\leqslant \int_0^T |x(\tau) - x_n(\tau)|\, |h(t-\tau)|\, d\tau .$$

Thus, from the Schwarz inequality,

$$|r(t) - r_n(t)|^2 \leqslant \int_0^T |x(\tau) - x_n(\tau)|^2\, d\tau \int_0^T h^2(t-\tau)\, d\tau$$

$$\leqslant \int_0^T |x(\tau) - x_n(\tau)|^2\, d\tau \int_{-\infty}^{\infty} h^2(t)\, dt .$$

However, by assumption

$$\int_{-\infty}^{\infty} h^2(t)\, dt < \infty$$

and Lemma 1(a) proved that

$$\lim_{n\to\infty} \int_0^T |x(\tau) - x_n(\tau)|^2 = 0 \quad .$$

Thus,

$$\lim_{n\to\infty} |r(t) - r_n(t)| = 0 \qquad 0 \leq t \leq T_1$$

and uniform convergence is obtained. Since

$$r_n(t) = \sum_{i=1}^{n} x_i h\varphi_i(t) = \sum_{i=1}^{n} \sqrt{\lambda_i}\, x_i \Theta_i(t)$$

it follows that

$$r(t) = \sum_{i=1}^{\infty} \sqrt{\lambda_i}\, x_i \Theta_i(t)$$

and therefore that

$$(r, \Theta_j)_{T_1} = \sum_{i=1}^{\infty} \sqrt{\lambda_i}\, x_i (\Theta_j, \Theta_i)_{T_1} \quad .$$

But

$$(\Theta_j, \Theta_i)_{T_1} = \frac{1}{\sqrt{\lambda_i \lambda_j}} (h\varphi_j, h\varphi_i)_{T_1} = \frac{1}{\sqrt{\lambda_i \lambda_j}} (\varphi_j, K\varphi_i) = \sqrt{\frac{\lambda_i}{\lambda_j}} (\varphi_j, \varphi_i) = \delta_{ij}$$

Thus,

$$r(t) = \sum_{i=1}^{\infty} (r, \Theta_i)_{T_1} \Theta_i(t)$$

and the Lemma is proved.

The following Lemma presents the pertinent facts relating to the representation of n(t) by the functions $\{\Theta_i(t)\}$.

Lemma 1(c).

Let the $\{\Theta_i(t)\}$ be as defined above. Then the additive Gaussian noise n(t) can be written as

$$n(t) = \sum_{i=1}^{\infty} n_i \Theta_i(t) + n_r(t) \qquad 0 \leq t \leq T_1$$

where

$$n_i = (n, \Theta_i)_{T_1}$$

$$\overline{n_i} = 0$$

$$\overline{n_i n_j} = \delta_{ij}$$

and the random processes $\sum_{i=1}^{\infty} n_i\Theta_i(t)$ and $n_r(t)$ are statistically independent.

Proof.†

Defining

$$n_r(t) = n(t) - \sum_{i=1}^{\infty} n_i\Theta_i(t)$$

it follows that n(t) can be represented in the form indicated. By assumption, n(t) is a zero-mean process. Therefore,

$$\overline{n}_i = \overline{(n, \Theta_i)_{T_1}} = (\overline{n}, \Theta_i)_{T_1} = 0$$

and

$$\overline{n_i n_j} = \overline{(n, \Theta_i)_{T_1} (n, \Theta_j)_{T_1}} = \int_o^{T_1}\int_o^{T_1} \overline{n(\tau)\, n(t)}\ \Theta_i(\tau)\ \Theta_j(t)\ d\tau\, dt$$

$$= \int_o^{T_1}\int_o^{T_1} \delta(\tau - t)\ \Theta_i(\tau)\ \Theta_j(t)\ d\tau\, dt = (\Theta_i, \Theta_j)_{T_1} = \delta_{ij} \quad .$$

(Note that unit variance is obtained here due to the normalization assumed in Fig. 1.) Next, let $n_s(t)$ be defined as

$$n_s(t) = \sum_{i=1}^{\infty} n_i\Theta_i(t) \qquad 0 \leqslant t \leqslant T_1$$

and define $\underline{n}_s$ and $\underline{n}_r$ by

$$\underline{n}_s = [n_s(t_1), n_s(t_2), \ldots, n_s(t_n)]$$

$$\underline{n}_r = [n_r(t'_1), n_r(t'_2), \ldots, n_r(t'_m)] \quad .$$

Then, the random processes $n_s(t)$ and $n_r(t)$ will be statistically independent if and only if the joint density function for $\underline{n}_s$ and $\underline{n}_r$ factors into a product of the individual density functions, that is, if

$$P(\underline{n}_s, \underline{n}_r) = P_1(\underline{n}_s)\ P_2(\underline{n}_r) \qquad \text{for all } \{t_i\} \text{ and } \{t'_i\} \quad .$$

† The following discussion might more aptly be called a plausibility argument than a proof since the series $\sum_{i=1}^{\infty} n_i\theta_i(t)$ does not converge and since n(t) is infinite bandwidth white noise for which time samples do not exist. However, this argument is of interest for several reasons: (a) it leads to a heuristically satisfying result, (b) the same result has been obtained by Price[70] in a more rigorous but considerably more involved derivation, and (c) it can be applied without apology to the colored noise problems considered later.

However, since all processes are Gaussian it can be readily shown that this factoring occurs when all terms in the joint covariance matrix of the form $\overline{n_s(t_i)\, n_r(t_j')}$ vanish. From the previous definitions

$$\overline{n_s(t)\, n_r(t')} = \overline{\left[\sum_i n_i \Theta_i(t)\right] \left[n(t') - \sum_i n_i \Theta_i(t')\right]}$$

$$= \sum_i \int_o^{T_1} \overline{n(\tau)\, n(t')}\, \Theta_i(\tau)\, \Theta_i(t)\, d\tau - \sum_i \sum_j \overline{n_i n_j} \Theta_i(t)\, \Theta_j(t')$$

$$= \sum_i \Theta_i(t')\, \Theta_i(t) - \sum_i \Theta_i(t')\, \Theta_i(t) = 0 \quad .$$

Combining the results of these three Lemmas, it follows that for any $x(t) \in \mathfrak{L}_2(0, T)$

$$x(t) = \sum_i x_i \varphi_i(t)$$

and

$$y(t) = \sum_i y_i \Theta_i(t) + n_r(t)$$

where

$$x_i = (x, \varphi_i)$$

and

$$y_i = (y, \Theta_i)_{T_1} = \sqrt{\lambda_i}\, x_i + n_i$$

$$n_i = (n, \Theta_i)_{T_1} \quad .$$

Thus, only the presence of the noise "remainder term" $n_r(t)$ prevents the direct use of the vector equation

$$\underline{y} = [H]\, \underline{x} + \underline{n} \quad .$$

It will be found in all of the succeeding analyses, however, that the statistical independence of the $n_s(t)$ and $n_r(t)$ processes would cause $n_r(t)$ to have no effect on probability of error. Thus, for the present work, $n_r(t)$ can be deleted from the received signal space without loss of generality. This leads to the desired vector space representation presented in the statement of the theorem. Q.E.D.

2. Basis Functions for Colored Noise and Infinite Observation Interval

This section specifies basis functions for colored noise and a doubly infinite observation interval. Since many portions of the proof of the following theorem are similar to the proof of Theorem 1, reference will be made to the previous work where possible. For physical, as well

as mathematical, reasons the analysis of this section assumes that the following condition is satisfied:

$$\int_{-\infty}^{\infty} \frac{|H(f)|^2}{N(f)}\, df < \infty \quad .$$

Theorem 2.

Let the noise of Fig. 1 have a spectral density N(f), define two symmetric functions K(t − s) and $K_1(t-s)$ by†

$$K(t-s) \triangleq \begin{cases} \int_{-\infty}^{\infty} \frac{|H(f)|^2}{N(f)} \exp[j\omega(t-s)]\, df & 0 \leq t, s \leq T \\ 0 & \text{elsewhere} \end{cases}$$

and

$$K_1(t-s) \triangleq \int_{-\infty}^{\infty} N(f)^{-1} \exp[j\omega(t-s)]\, dt$$

and define a set of eigenfunctions and eigenvalues by

$$\lambda_i \varphi_i(t) = K\varphi_i(t) \qquad \begin{cases} 0 \leq t \leq T \\ i = 1, 2, \ldots \end{cases} \quad . \tag{10}$$

Then the vector representation of an arbitrary $x(t) \in \mathfrak{L}_2(0,T)$ is $\underline{x}$, where $x_i = (x, \varphi_i)$ and the vector representation of the corresponding y(t) on the doubly infinite interval $[-\infty, \infty]$ is given by Eq. (7) in which [H] and $\underline{n}$ have the same properties as in Theorem 1. The basis functions for y(t) are

$$\Theta_i(t) = \begin{cases} \frac{1}{\sqrt{\lambda_i}} h\varphi_i(t) & 0 \leq t < \infty \\ 0 & t < 0 \end{cases}$$

and the i^{th} component of $\underline{y}$ is

$$y_i = \langle y, K_1\Theta_i \rangle \triangleq \int_{-\infty}^{\infty} y(t)\, K_1\Theta_i(t)\, dt \quad .$$

Proof.

Under the conditions assumed for this theorem, the functions $\{\varphi_i(t)\}$ of Eq. (10) are simply a special case of Eq. (8) with $T_1 = +\infty$. Thus, they form a basis for $\mathfrak{L}_2(0,T)$ and the representation for x(t) follows directly. [See Lemma 1(a) for a proof of this result and a discussion of the convergence obtained.] By defining $\Theta_i(t)$ as it was previously defined, it follows directly from Lemma 1(b) that the filter output r(t) is given by

†Because N(f) is an arbitrary spectral density, it is possible that $N(f)^{-1}$ will be unbounded for large f and therefore that the integral defining $K_1(t-s)$ will not exist in a strict sense. It will be observed, however, that $K_1(t-s)$ is always used under an integral sign in such a manner that the over-all operation is physically meaningful. It should be noted that the detection of a known signal in the presence of colored Gaussian noise involves an identical operation.[54]

$$r(t) = \sum_{i=1}^{\infty} \sqrt{\lambda_i}\; x_i \Theta_i(t) \qquad 0 \leq t < \infty \quad .$$

Furthermore, since

$$\langle \Theta_i, K_1 \Theta_j \rangle = \frac{1}{\sqrt{\lambda_i \lambda_j}} \langle h\varphi_i, K_1 h\varphi_j \rangle$$

$$= \frac{1}{\sqrt{\lambda_i \lambda_j}} \int_{-\infty}^{\infty} \left[\int_0^T \varphi_i(\sigma)\, h(t-\sigma)\, d\sigma \right] \left[\int_{-\infty}^{\infty} K_1(t-s) \int_0^T \varphi_j(\rho)\, h(s-\rho)\, d\rho\, ds \right] dt$$

$$= \frac{1}{\sqrt{\lambda_i \lambda_j}} \int_0^T \varphi_i(\sigma) \int_0^T \varphi_j(\rho) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(t-\sigma)\, K_1(t-s)\, h(s-\rho)\, dt\, ds\, d\sigma\, d\rho$$

$$= \frac{1}{\sqrt{\lambda_i \lambda_j}} \int_0^T \varphi_i(\sigma) \int_0^T \varphi_j(\rho) \int_{-\infty}^{\infty} \frac{|H(f)|^2}{N(f)} \exp[j\omega(\sigma - \rho)]\, df\, d\sigma\, d\rho$$

$$= \frac{1}{\sqrt{\lambda_i \lambda_j}} (\varphi_i, K\varphi_j)_T = \sqrt{\frac{\lambda_j}{\lambda_i}} (\varphi_i, \varphi_j) = \delta_{ij}$$

it follows that

$$\langle r, K_1 \Theta_i \rangle = \sqrt{\lambda_i}\; x_i$$

and thus that

$$r(t) = \sum_{i=1}^{\infty} \langle r, K_1 \Theta_i \rangle\, \Theta_i(t) \quad .$$

Finally, with $\mathcal{R}_n(\tau)$ and n_i defined by

$$\mathcal{R}_n(\tau) \triangleq \int_{-\infty}^{\infty} N(f)\, e^{j\omega\tau}\, df$$

$$n_i \triangleq \langle n, K_1 \Theta_i \rangle$$

it follows that

$$n_i = \langle \overline{n}, K_1 \Theta_i \rangle = 0$$

and

$$\overline{n_i n_j} = \overline{\langle n, K_1 \Theta_i \rangle \langle n, K_1 \Theta_j \rangle}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{R}_n(t-s) \int_{-\infty}^{\infty} K_1(t-\sigma)\, \Theta_i(\sigma)\, d\sigma \int_{-\infty}^{\infty} K_1(s-\rho)\, \Theta_j(\rho)\, d\rho\, dt\, ds$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Theta_i(\sigma)\, \Theta_j(\rho)\, d\sigma\, d\rho \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{R}_n(t-s)\, K_1(t-\sigma)\, K_1(s-\rho)\, ds\, dt$$

or

$$\overline{n_i n_j} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Theta_i(\sigma)\, \Theta_j(\rho)\, K_1(\sigma - \rho)\, d\sigma\, d\rho = \langle \Theta_i, K_1 \Theta_j \rangle = \delta_{ij} \quad .$$

From this it is readily shown, following a procedure identical to that used in Lemma 1(c), that the random processes $n_s(t)$ and $n_r(t)$ defined by

$$n_s(t) \triangleq \sum_{i=1}^{\infty} n_i \Theta_i(t)$$

$$n_r(t) \triangleq n(t) - n_s(t)$$

are statistically independent. Thus, neglecting the "remainder term" leads to

$$y(t) = \sum_{i=1}^{\infty} y_i \Theta_i(t)$$

where

$$y_i = \langle y, K_1 \Theta_i \rangle = \sqrt{\lambda_i}\, x_i + n_i$$

or, in vector notation,

$$\underline{y} = [H]\, \underline{x} + \underline{n} \quad .$$

3. Basis Functions for Colored Noise and Finite Observation Interval

This section specifies basis functions for colored noise and an arbitrary, finite observation interval. As in Sec. II-D-2, it is assumed that

$$\int_{-\infty}^{\infty} \frac{|H(f)|^2}{N(f)}\, df < \infty \quad .$$

Theorem 3.

Let the noise of Fig. 1 have a spectral density $N(f)$, define a function† $K_1(t, s)$ by

$$\int_0^{T_1} \mathcal{R}_n(t - \sigma)\, K_1(\sigma, s)\, d\sigma = \delta(t - s) \qquad 0 \leqslant t, s \leqslant T_1$$

where

$$\mathcal{R}_n(\tau) \triangleq \int_{-\infty}^{\infty} N(f)\, e^{j\omega\tau}\, df$$

define a function $K(t, s)$ by

$$K(t, s) \triangleq \begin{cases} \displaystyle\int_0^{T_1} \int_0^{T_1} h(\sigma - t)\, K_1(\sigma, \rho)\, h(\rho - s)\, d\sigma\, d\rho & 0 \leqslant t, s \leqslant T \\ 0 & \text{elsewhere} \end{cases}$$

† Comments identical to those in the footnote to Theorem 2 also apply to this "function."

and define a set of eigenfunctions and eigenvalues by

$$\lambda_i \varphi_i(t) = K\varphi_i(t) \qquad \begin{cases} 0 \leqslant t \leqslant T \\ i = 1, 2, \ldots \end{cases} \tag{11}$$

Then the vector representation of an arbitrary $x(t) \in \mathcal{L}_2(0, T)$ is $\underline{x}$, where $x_i = (x, \varphi_i)$ and the vector representation of the corresponding $y(t)$ on the interval $[0, T_1]$ is given by Eq. (7) in which [H] and $\underline{n}$ have the same properties as in Theorem 1. The basis functions for $y(t)$ are

$$\Theta_i(t) = \begin{cases} \dfrac{1}{\sqrt{\lambda_i}} h\varphi_i(t) & 0 \leqslant t \leqslant T_1 \\ 0 & \text{elsewhere} \end{cases}$$

and the i^{th} component of $\underline{y}$ is

$$y_i = (y, K_1\Theta_i)_{T_1} \quad .$$

Proof.

This proof consists of a series of Lemmas. The first Lemma presents an interesting property of a complete set of orthonormal functions.

Lemma 3(a).

Let $\{\gamma_i(t)\}$ be an arbitrary set of orthonormal functions that form a basis for $\mathcal{L}_2(0, T_1)$, i.e., they are complete in $\mathcal{L}_2(0, T_1)$. Then

$$\sum_{i=1}^{\infty} \gamma_i(t)\, \gamma_i(s) = \delta(t - s) \qquad 0 \leqslant t, s \leqslant T_1$$

in the sense that for any $f(t) \in \mathcal{L}_2(0, T_1)$

$$\underset{n\to\infty}{\text{l.i.m.}} \int_0^{T_1} f(\sigma) \left[\sum_{i=1}^{n} \gamma_i(t)\, \gamma_i(\sigma)\right] d\sigma = f(t) \quad .$$

Proof.

By definition, a set of orthonormal functions $\{\gamma_i(t)\}$ that are complete in $\mathcal{L}_2(0, T_1)$ have the property that for any $f(t) \in \mathcal{L}_2(0, T_1)$

$$f(t) = \underset{n\to\infty}{\text{l.i.m.}} \sum_{i=1}^{n} (f, \gamma_i)_{T_1}\, \gamma_i(t) \qquad 0 \leqslant t \leqslant T_1$$

that is,

$$f(t) = \underset{n\to\infty}{\text{l.i.m.}} \int_0^{T_1} f(\sigma) \left[\sum_{i=1}^{n} \gamma_i(\sigma)\, \gamma_i(t)\right] d\sigma \quad .$$

The following Lemma provides a constructive proof that the function $K_1(t, s)$ defined in the theorem exists.

Lemma 3(b).

Define a function $K_n(t-s)$ by

$$K_n(t-s) \triangleq \begin{cases} \mathcal{R}_n(t-s) & 0 \leqslant t, s \leqslant T_1 \\ 0 & \text{elsewhere} \end{cases}$$

and define a set of eigenfunctions $\{\gamma_i(t)\}$ and eigenvalues $\{\beta_i\}$ by

$$\beta_i \gamma_i(t) = K_n \gamma_i(t) \qquad \begin{cases} 0 \leqslant t \leqslant T_1 \\ i = 1, 2, \ldots \end{cases} .$$

Then

$$K_1(t, s) = \sum_{i=1}^{\infty} \frac{\gamma_i(t)\, \gamma_i(s)}{\beta_i} \qquad 0 \leqslant t, s \leqslant T_1 \quad .$$

Proof.

From Mercer's theorem[64] it is known that

$$\mathcal{R}_n(t-s) = \sum_{i=1}^{\infty} \beta_i \gamma_i(t)\, \gamma_i(s) \qquad 0 \leqslant t, s \leqslant T_1 \quad .$$

Thus, with

$$K_1(t, s) = \sum_{i=1}^{\infty} \frac{\gamma_i(t)\, \gamma_i(s)}{\beta_i} \qquad 0 \leqslant t, s \leqslant T_1$$

it follows that

$$\int_0^{T_1} \mathcal{R}_n(t-\sigma)\, K_1(\sigma, s)\, d\sigma = \sum_i \sum_j \frac{\beta_j}{\beta_i} \gamma_j(t)\, \gamma_i(s)\, (\gamma_j, \gamma_i)_{T_1}$$

$$= \sum_i \gamma_i(t)\, \gamma_i(s) \quad .$$

This result, together with Lemma 3(a) and the known[61] fact that the $\{\gamma_i(t)\}$ are complete, finishes the proof.

Lemma 3(c).

If $T_1 \geqslant T$, the $\{\varphi_i(t)\}$ defined by Eq.(11) form an orthonormal basis for $\mathcal{L}_2(0, T)$, that is, for every $x(t) \in \mathcal{L}_2(0, T)$

$$x(t) = \sum_{i=1}^{\infty} x_i \varphi_i(t) \qquad 0 \leqslant t \leqslant T$$

where

$x_i = (x, \varphi_i)$ and convergence is mean square.

Proof.

Since $K(t, s)$ is an $\mathcal{L}_2$-kernel,† it follows from the proof of Lemma 1(a) that the $\{\varphi_i(t)\}$ are complete in $\mathcal{L}_2(0, T)$ if and only if the condition

$$(f, Kf)_T = 0$$

implies $f(t) = 0$ almost everywhere on $[0, T]$. But

$$(f, Kf)_T = \int_0^{T_1} \int_0^{T_1} \int_0^T \int_0^T f(t)\, h(\sigma - t)\, K_1(\sigma, \rho)\, h(\rho - s)\, f(s)\, dt\, ds\, d\sigma\, d\rho$$

Thus, from Lemma 3(b),

$$(f, Kf)_T = \sum_i \frac{1}{\beta_i} \left[\int_0^{T_1} \int_0^T f(t)\, h(\sigma - t)\, \gamma_i(\sigma)\, dt\, d\sigma\right]^2$$

and it follows that the condition

$$(f, Kf)_T = 0$$

implies

$$\int_0^{T_1} \left[\int_0^T f(t)\, h(\sigma - t)\, dt\right] \gamma_i(\sigma)\, d\sigma = 0 \qquad i = 1, 2, \ldots \quad .$$

However, the completeness of the $\gamma_i(t)$ implies that the only function orthogonal to all the $\gamma_i(t)$ is the function that is zero almost everywhere.[65] Thus $(f, Kf)_T = 0$ implies

$$\int_0^T f(t)\, h(\sigma - t)\, dt = 0 \qquad \text{almost everywhere on } [0, T_1]$$

and, from the proof of Lemma 1(a), it follows that, for $T_1 \geqslant T$, $f(t) = 0$ almost everywhere on $[0,T]$.

This finishes the proof that the $\{\varphi_i(t)\}$ are complete. The following paragraph outlines a proof of the remainder of the theorem.

With $\Theta_i(t)$ as previously defined, it follows directly from Lemmas 1(b) and 3(c) that for any $x(t) \in \mathcal{L}_2(0, T)$ the filter output $r(t)$ is given by

$$r(t) = \sum_{i=1}^{\infty} \sqrt{\lambda_i}\, x_i \Theta_i(t) \qquad 0 \leqslant t \leqslant T_1 \quad .$$

Furthermore, by a procedure identical to that in Theorem 2, it follows that

$$(\Theta_i, K_1\Theta_j)_{T_1} = \frac{1}{\sqrt{\lambda_i \lambda_j}} (\varphi_i, K\varphi_j)_T = \delta_{ij}$$

which implies that

$$(r, K_1\Theta_i)_{T_1} = \sqrt{\lambda_i}\, x_i$$

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† See Appendix A.

and thus that

$$r(t) = \sum_{i=1}^{\infty} (r, K_1\Theta_i)_{T_1} \Theta_i(t) \quad .$$

Next, with n_i defined by

$$n_i = (n, K_1\Theta_i)_{T_1}$$

it follows that

$$\overline{n}_i = (\overline{n}, K_1\Theta_i)_{T_1} = 0$$

and, by a procedure identical to that in Theorem 2, that

$$\overline{n_i n_j} = \overline{(n, K_1\Theta_i)_{T_1} (n, K_1\Theta_j)_{T_1}} = (\Theta_i, K_1\Theta_j)_{T_1} = \delta_{ij} \quad .$$

From this it is readily shown, following a procedure identical to that in Lemma 1(c), that the random processes $n_s(t)$ and $n_r(t)$ defined by

$$n_s(t) \triangleq \sum_{i=1}^{\infty} n_i\Theta_i(t) \qquad 0 \leqslant t \leqslant T_1$$

$$n_r(t) \triangleq n(t) - n_s(t) \qquad 0 \leqslant t \leqslant T_1$$

are statistically independent. Thus, neglect of the "remainder term" leads to

$$y(t) = \sum_{i=1}^{\infty} y_i\Theta_i(t) \qquad 0 \leqslant t \leqslant T_1$$

where

$$y_i = (y, K_1\Theta_i)_{T_1} = \sqrt{\lambda_i}\, x_i + n_i$$

or, in vector notation

$$\underline{y} = [H]\, \underline{x} + \underline{n}$$

and the theorem is proved.

4. Interpretation of Signal Representations

The previous sections have presented several results concerning signal representation for the channel of Fig. 1. Since these results have of necessity been presented in a highly mathematical context, it is of interest to interpret these in terms of more physical engineering concepts; in particular, it is desirable to interpret these in terms of optimum detectors for digital communication.

Consider first the $\{\varphi_i(t)\}$ of Theorem 1. The first and foremost property of these functions is that they are solutions of the integral equation

$$\lambda_i\varphi_i(t) = \int_0^T \varphi_i(\tau)\, K(t, \tau)\, d\tau \qquad 0 \leqslant t \leqslant T \tag{12}$$

where

$$K(t,s) = \int_0^{T_1} h(\sigma - t)\, h(\sigma - s)\, d\sigma \quad .$$

To understand the physical meaning of this relation, consider the optimum detector for white noise and an observation interval $[0, T_1]$ when a time-limited signal x(t) is transmitted. If r(t) is the corresponding channel filter output on $[0, T_1]$, it is well known that the optimum detector cross correlates r(t) with the channel output over the interval $[0, T_1]$. Interpreted in terms of linear time-invariant filters,† this operation can be performed as shown in Fig. 2 where the matched filter has been realized as the cascade of a filter matched to the channel filter over the interval $[0, T_1]$ followed by a multiply-and-integrate operation. The significance of the fact that the $\{\varphi_i(t)\}$ are a solution to Eq. (12) is now made clear by noting that the (time-variant) impulse response of the cascade of the channel filter and the "channel portion" of the matched filter is

$$K(t,s) = \int_0^{T_1} h(\sigma - t)\, h(\sigma - s)\, d\sigma \qquad 0 \leqslant t, s \leqslant T \tag{13}$$

where K(t, s) is the response at time t to an impulse applied at time s. Thus, the $\{\varphi_i(t)\}$ are simply a set of signals that are self-reproducing (to within a gain factor λ_i) over the interval [0, T] when passed through the filter K(t, s).

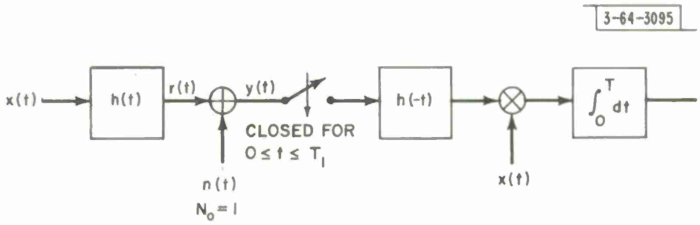

Fig. 2. Concerning interpretation of the $\{\phi_i(t)\}$ of Theorem 1.

This feature or, more basically, the fact that the $\{\varphi_i(t)\}$ are a solution to Eq. (12), causes the $\{\varphi_i(t)\}$ to have two extremely important properties. This first property, that the $\{\varphi_i(t)\}$ are orthogonal over the interval [0, T] and may be assumed normalized, is readily shown in the following manner. Assume that for $i \neq j$, $\lambda_i \neq \lambda_j$.‡ Then it follows that

$$\int_0^T \varphi_i(t)\, \lambda_j \varphi_j(t)\, dt = \int_0^T \varphi_i(t) \int_0^T \varphi_j(\tau)\, K(t,\tau)\, d\tau\, dt$$

or

$$\lambda_j(\varphi_i, \varphi_j) = (\varphi_i, K\varphi_j) \tag{14}$$

and similarly since K(t, s) = K(s, t),

$$\int_0^T \varphi_j(t)\, \lambda_i \varphi_i(t)\, dt = \int_0^T \varphi_j(t) \int_0^T \varphi_i(\tau)\, K(t,\tau)\, d\tau\, dt$$

† Here, and throughout this work, the question of the physical realizability of all filters other than the channel filter has been ignored.

‡ It can be shown via an argument too long to present here that all $\phi_i(t)$ having a common eigenvalue can be assumed to be orthogonal.[64,65]

$$\int_0^T \varphi_j(t)\, \lambda_i \varphi_i(t)\, dt = \int_0^T \varphi_i(\tau) \int_0^T \varphi_j(t)\, K(\tau, t)\, dt\, d\tau$$

or

$$\lambda_i(\varphi_i, \varphi_j) = (\varphi_i, K\varphi_j) \quad . \tag{15}$$

Therefore, upon subtracting Eq. (14) from Eq. (15) it follows that

$$(\lambda_j - \lambda_i)\, (\varphi_i, \varphi_j) = 0 \quad .$$

But, by assumption, $(\lambda_j - \lambda_i) \neq 0$. Thus $(\varphi_i, \varphi_j) = 0$ and the $\{\varphi_i(t)\}$ are orthogonal. The fact that the $\{\varphi_i(t)\}$ can be assumed to be normalized follows directly from Eq. (12) by observing that if $\varphi_i(t)$ is a solution to this equation, then $c\varphi_i(t)$ is also a solution when c is an arbitrary constant.

The second property, that the $\{\varphi_i(t)\}$ are orthogonal over the interval $[0, T_1]$ after passing through the filter, follows in a straightforward manner. Let $r_j(t)$ be the channel filter output when $\varphi_j(t)$ is transmitted. Then

$$\begin{aligned}
\int_0^T r_i(t)\, r_j(t)\, dt &= \int_0^{T_1} \int_0^T \varphi_i(\sigma)\, h(t-\sigma)\, d\sigma \int_0^T \varphi_j(\rho)\, h(t-\rho)\, d\rho\, dt \\
&= \int_0^T \int_0^T \varphi_i(\sigma)\, \varphi_j(\rho) \int_0^{T_1} h(t-\sigma)\, h(t-\rho)\, dt\, d\rho\, d\sigma \\
&= \int_0^T \varphi_i(\sigma) \left[\int_0^T \varphi_j(\rho)\, K(\sigma, \rho)\, d\rho\right] d\sigma \\
&= \lambda_j \int_0^T \varphi_i(\sigma)\, \varphi_j(\sigma)\, d\sigma = \lambda_j(\varphi_i, \varphi_j) = \lambda_j \delta_{ij} \quad .
\end{aligned} \tag{16}$$

Thus, the $\{\varphi_i(t)\}$ are orthogonal after passing through the filter. This property is important because it allows the channel memory (its time-dispersive characteristic) to be treated analytically in terms of a number of parallel and independent time-discrete channels with different gains. In other words, when the transmitted signal is written as

$$x(t) = \sum_i x_i \varphi_i(t)$$

it follows from Theorem 1 that the received signal can be written as

$$y(t) = \sum_i y_i \Theta_i(t)$$

where

$$\Theta_i(t) \triangleq \frac{1}{\sqrt{\lambda_i}} \int_0^T \varphi_i(\tau)\, h(t-\tau)\, d\tau$$

and, due to the fact that the $\Theta_i(t)$ are orthonormal as just demonstrated $[\sqrt{\lambda_i}\, \Theta_i(t) = r_i(t)]$,

$$y_i = (y, \Theta_i)_{T_1} = \sqrt{\lambda_i}\, x_i + n_i \tag{17}$$

with

$$\overline{n_i n_j} = \delta_{ij} \quad .$$

This is simply a statement that to obtain $\underline{y}$ each coordinate of $\underline{x}$ is passed through an independent time-discrete additive Gaussian channel with gain $\sqrt{\lambda_i}$ and unity noise variance. Thus, for purposes of analysis, the channel of Fig. 1 simplifies to that of Fig. 3.

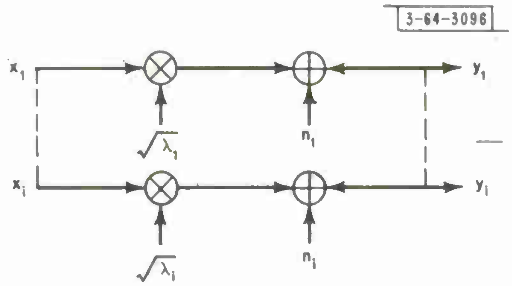

Fig. 3. Mathematical equivalent of channel in Fig. 1.

Finally, it is of interest to interpret the inner product for y_i in terms of Fig. 2. From Eq. (17), y_i is given by

$$y_i = (y, \Theta_i)_{T_1} = \int_0^{T_1} y(t)\, \Theta_i(t)\, dt$$

$$= \int_0^{T_1} y(t) \int_0^{T} \frac{\varphi_i(\tau)}{\sqrt{\lambda_i}}\, h(t-\tau)\, d\tau\, dt$$

$$= \frac{1}{\sqrt{\lambda_i}} \int_0^{T} \varphi_i(\tau) \int_0^{T_1} y(t)\, h(t-\tau)\, dt\, d\tau \quad . \tag{18}$$

Comparing Eq. (18) with the filtering operations indicated in Fig. 2 shows that y_i is precisely $1/\sqrt{\lambda_i}$ times the output of the optimum (matched-filter) detector when $\varphi_i(t)$ is transmitted. This result becomes important in practice when a large number ($M >> d$) of d-dimensional signals are to be transmitted, since the construction of d "coordinate filters" is much simpler than the construction of M different matched filters.

The previous discussion has shown that in spite of the highly mathematical nature of the signal representation of Theorem 1, it is possible to readily understand the important properties of the $\{\varphi_i(t)\}$ by interpreting them in terms of optimum detectors. An analogous discussion of the properties of the $\{\varphi_i(t)\}$ of Theorem 2 follows.

As before, the $\{\varphi_i(t)\}$ are defined as solutions to the integral equation

$$\lambda_i \varphi_i(t) = \int_0^{T} \varphi_i(\tau)\, K(t,\tau)\, d\tau \qquad 0 \leqslant t \leqslant T \tag{19}$$

where $K(t,s)$ is now given by

$$K(t,s) = K(t-s) = \int_{-\infty}^{\infty} \frac{|H(f)|^2}{N(f)} \exp[j\omega(t-s)]\, df \quad .$$

To obtain a physical understanding of Eq. (19), consider the optimum detector for colored noise and a doubly infinite observation interval when a time-limited signal x(t) is transmitted. It is

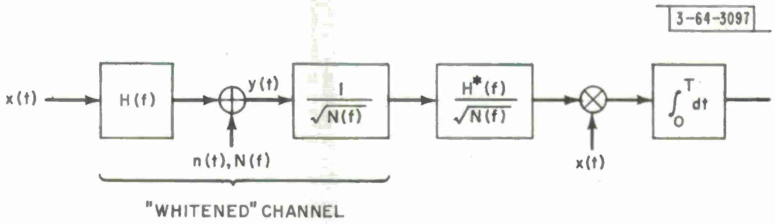

Fig. 4. Concerning interpretation of the $\{\phi_i(t)\}$ of Theorem 2.

known[54] that the optimum detector cross correlates the channel output y(t) with a function q(t) over the doubly infinite interval, where

$$q(t) = \int_{-\infty}^{\infty} \frac{X(f)\,H(f)}{N(f)}\, e^{j\omega t}\, df \quad .$$

Interpreted in terms of linear time-invariant filters, this operation can be performed as shown in Fig. 4. Note that the optimum detector has been formed here as the cascade of a "prewhitening" filter, a filter matched to the "whitened" channel filter characteristic and a multiply-and-integrate operation. Upon observing that the impulse response of the cascade of the channel filter and the "channel portion" of the optimum detector is given by

$$K(t) = \int_{-\infty}^{\infty} \frac{|H(f)|^2}{N(f)}\, e^{j\omega t}\, df$$

it follows from Eq. (19) that the $\{\varphi_i(t)\}$ are simply a set of signals that are self-reproducing (to within a gain factor λ_i) over the interval [0, T] when passed through the filter K(t). As before, this characteristic causes the $\{\varphi_i(t)\}$ to have two important properties. The first property, that the $\{\varphi_i(t)\}$ are orthonormal, follows directly from the discussion of the $\{\varphi_i(t)\}$ of Eq. (12). The second property, that the $\{\varphi_i(t)\}$ are orthogonal at the channel output with respect to a generalized inner product, has been demonstrated in the proof of Theorem 2. A more physical interpretation of the latter property can be obtained by investigating the orthogonality of the $\varphi_i(t)$ at the output of the "whitened" channel. Let $r_i(t)$ be the output of this channel when $\varphi_i(t)$ is transmitted; let $\Theta_i(t)$ and $K_1(t)$ be defined as in Theorem 2, i.e.,

$$\Theta_i(t) = \frac{1}{\sqrt{\lambda_i}} \int_0^T \varphi_i(\tau)\, h(t-\tau)\, d\tau$$

and

$$K_1(t) = \int_{-\infty}^{\infty} \frac{1}{N(f)}\, e^{j\omega t}\, df$$

and let $h_w(t)$ be the impulse response of the "prewhitening" filter. Then

$$\int_{-\infty}^{\infty} r_j(t)\, r_i(t)\, dt = \sqrt{\lambda_i \lambda_j} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \Theta_i(\sigma)\, h_w(t-\sigma)\, d\sigma\right] \left[\int_{-\infty}^{\infty} \Theta_j(\rho)\, h_w(t-\rho)\, d\rho\right] dt$$

$$= \sqrt{\lambda_i \lambda_j} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Theta_i(\sigma)\, \Theta_j(\rho) \int_{-\infty}^{\infty} h_w(t-\sigma)\, h_w(t-\rho)\, dt\, d\sigma\, d\rho$$

$$\int_{-\infty}^{\infty} r_j(t)\, r_i(t)\, dt = \sqrt{\lambda_i \lambda_j} \int_{-\infty}^{\infty} \Theta_i(\sigma) \int_{-\infty}^{\infty} \Theta_j(\rho)\, K_1(\sigma - \rho)\, d\rho\, d\sigma$$

$$= \sqrt{\lambda_i \lambda_j} \left\langle \Theta_i, K_1 \Theta_j \right\rangle = \lambda_i \delta_{ij}$$

where the last line follows from the proof of Theorem 2. Thus, the statement that the $\varphi_i(t)$ are orthogonal at the channel output with respect to a generalized inner product is equivalent to the statement that the $\{\varphi_i(t)\}$ are orthogonal at the output of the "whitened" channel.

From these orthogonality properties it follows as shown in Theorem 2, that if x(t) is written as

$$x(t) = \sum_i x_i \varphi_i(t)$$

then y(t) can be written as

$$y(t) = \sum_i y_i \Theta_i(t)$$

where

$$y_i = \left\langle y, K_1 \Theta_i \right\rangle \tag{20}$$

and

$$\overline{n_i n_j} = \delta_{ij} \quad .$$

Thus, the orthogonality properties of the $\{\varphi_i(t)\}$ again allow the channel memory (its time-dispersive characteristic and the colored noise) to be treated analytically in terms of a number of parallel and independent channels with different gains as shown in Fig. 3.

Finally, it is of interest to interpret the inner product for y_i in terms of Fig. 4. From Eq. (20), y_i is given by

$$y_i = \left\langle y, K_1 \Theta_i \right\rangle = \int_{-\infty}^{\infty} y(t) \int_{-\infty}^{\infty} K_1(t - \tau)\, \Theta_i(\tau)\, d\tau\, dt$$

$$= \int_{-\infty}^{\infty} y(t) \int_{0}^{T} \frac{\varphi_i(\sigma)}{\sqrt{\lambda_i}} \int_{-\infty}^{\infty} h(\tau - \sigma)\, K_1(t - \tau)\, d\tau\, d\sigma\, dt$$

$$= \frac{1}{\sqrt{\lambda_i}} \int_{0}^{T} \varphi_i(\sigma) \int_{-\infty}^{\infty} y(t) \int_{-\infty}^{\infty} \frac{H^*(f)}{N(f)} \exp[j\omega(\sigma - t)]\, df\, dt\, d\sigma \quad . \tag{21}$$

Comparing Eq. (21) with the filtering operations indicated in Fig. 4 shows that y_i is precisely $1/\sqrt{\lambda_i}$ times the output of the optimum detector when $\varphi_i(t)$ is transmitted. It is interesting to note that this is identical to the result obtained previously for the $\{\varphi_i(t)\}$ of Theorem 1.

In conclusion, it should be mentioned that the $\{\varphi_i(t)\}$ of Theorem 3 can also be interpreted in terms of the appropriate optimum detector in a manner directly analogous to the preceding discussion. In this case, however, the derivation of the (time-variant) "prewhitening" filter

and the remainder of the optimum detector becomes as mathematically involved as was the proof of the Theorem.

5. Some Optimum Signal Properties of Basis Functions

The functions $\{\varphi_i(t)\}$ have some additional properties that are of interest from the standpoint of optimum signal design. Consider the situation in which one of two time-limited signals is to be transmitted through the channel of Fig. 1. Let these signals be $\pm x(t)$. It is desired to select the signal waveform x(t) so that the probability of error is minimized for fixed signal energy when an optimum detector is used. Depending upon the noise spectral density and the receiver observation interval, the following results are obtained.†

a. White Noise, Arbitrary Observation Interval $[0, T_1]$

It is well known that the optimum detector (matched filter) for this situation makes a decision based upon the quantity

$$\int_0^{T_1} y(t)\, r(t)\, dt = (y, hx)_{T_1} = \underline{y} \cdot \underline{r}$$

where $\underline{r} = [H]\, \underline{x}$ denotes the usual vector inner product, and the basis functions, with respect to which $\underline{x}$, $\underline{y}$, and $\underline{r}$ are defined, are those of Theorem 1. The probability of error for this detector is given by

$$P_e = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\sqrt{E/N_o}} \exp[-1/2\, t^2]\, dt$$

where

$$E = (hx, hx)_{T_1} = \underline{r} \cdot \underline{r} = \sum_{i=1}^{\infty} \lambda_i x_i^2$$

and N_o is the (double-sided) noise spectral density. Thus, since the transmitted signal energy is

$$(x, x) = \underline{x} \cdot \underline{x} = \sum_{i=1}^{\infty} x_i^2$$

and since by convention (Ref. 64) $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \dots$, it follows that for fixed input signal energy the output signal energy is maximum (and therefore the probability of error is minimum) when $x(t) = \varphi_1(t)$.‡ More generally, it is quite easily shown from these results that $x(t) = \varphi_j(t)$ is the signal giving maximum output energy on the interval $[0, T_1]$ under the constraints

$$(x, \varphi_i) = 0 \qquad i = 1, 2, \dots, j-1$$

$$(x, x) = 1$$

† Note that these results assume either a single transmission or negligible intersymbol interference.

‡ Note that $\lambda_i = (h\phi_i, h\phi_i)_{T_1} / (\phi_i, \phi_i)$. Thus, λ_i is effectively an energy "transfer ratio" and $\phi_1(t)$ is the $\mathcal{L}_2(0, T)$ signal having the largest "transfer ratio," i.e., it is "attenuated" least in passing through the filter. This property appears to have been first recognized by Chalk[71] for the special case $T_1 = \infty$.

At this point, two additional properties of the $\{\varphi_i(t)\}$ of Theorem 1 should be mentioned:

(1) When $T_1 = \infty$, Eq. (7) reduces to the well-known[60] integral equation involved in the Karhunen-Loeve expansion of a random process with autocorrelation function $\mathcal{R}_n(t-s) = K(t,s)$;

(2) When K(t, s) is defined as

$$K(t,s) \triangleq \begin{cases} \int_{-\infty}^{\infty} h(\sigma - t)\, h(\sigma - s)\, d\sigma & 0 \leqslant t, s \leqslant T \\ 0 & \text{elsewhere} \end{cases}$$

and h(t) is specialized to $h(t) = (\sin \pi t)/\pi t$, the $\{\varphi_i(t)\}$ are the prolate spheroidal wave functions studied by Slepian, Pollak, and Landau.[72,73]

The next section demonstrates that $\varphi_1(t)$ of Theorem 2 is the optimum signal for binary transmission when the noise is colored and a doubly infinite observation interval is used.

b. Colored Noise, Infinite Observation Interval

The optimum detector for this situation is known[54] to make a decision based on the quantity

$$\int_{-\infty}^{\infty} y(t)\, q(t)\, dt$$

where

$$q(t) = \int_{-\infty}^{\infty} \frac{X(f)\, H(f)}{N(f)}\, e^{j\omega t}\, df \quad .$$

Introducing the signal representation of Theorem 2, this becomes

$$\int_{-\infty}^{\infty} y(t)\, q(t)\, dt = \langle y, K_1 r \rangle = \underline{y} \cdot \underline{r} = \underline{y} \cdot [H]\, \underline{x} \quad .$$

The probability of error for this case is determined by the quantity

$$\int_{-\infty}^{\infty} \frac{|X(f)\, H(f)|^2}{N(f)}\, df = \langle r, K_1 r \rangle = \underline{r} \cdot \underline{r} = \sum_{i=1}^{\infty} \lambda_i x_i^2$$

which might aptly be called the "generalized E/N_o." Thus, since the input energy is $(x,x) = \underline{x} \cdot \underline{x}$, it follows that again $x(t) = \varphi_1(t)$ is the optimum signal to be used to minimize probability of error for fixed input signal energy.

In conclusion, the following section demonstrates that $\varphi_1(t)$ of Theorem 3 is the optimum signal for binary transmission when the noise is colored and a finite observation interval $[0, T_1]$ is used.

c. Colored Noise, Finite Observation Interval

For this situation, the optimum detector is known[54] to make a decision based on the quantity

$$\int_{0}^{T_1} y(t)\, q(t)\, dt$$

where

$$q(t) \triangleq \sum_{i=1}^{\infty} \frac{(r, \gamma_i)_{T_1}}{\beta_i} \gamma_i(t) \qquad 0 \leqslant t \leqslant T_1$$

with

$$\beta_i \gamma_i(t) = K_n \gamma_i(t) \qquad 0 \leqslant t \leqslant T_1$$

and

$$K_n(t-s) \triangleq \begin{cases} \int_{-\infty}^{\infty} N(f) \exp[j\omega(t-s)]\, df & 0 \leqslant t, s \leqslant T_1 \\ 0 & \text{elsewhere} \end{cases} .$$

In terms of the signal representation of Theorem 3, this becomes

$$\int_0^{T_1} y(t)\, q(t)\, dt = \int_0^{T_1} y(t) \left[\sum_{i=1}^{\infty} \frac{(r, \gamma_i)_{T_1}}{\beta_i} \gamma_i(t) \right] dt$$

$$= \int_0^{T_1} y(t) \left\{ \int_0^{T_1} r(s) \left[\sum_{i=1}^{\infty} \frac{\gamma_i(s)\, \gamma_i(t)}{\beta_i} \right] ds \right\} dt$$

$$= (y, K_1 r)_{T_1} = \sum_{i,j} y_i \sqrt{\lambda_j}\, x_j (\Theta_i, K_1 \Theta_j)_{T_1}$$

$$= \sum_i y_i \sqrt{\lambda_j}\, x_j = \underline{y} \cdot [H]\, \underline{x} = \underline{y} \cdot \underline{r} \quad .$$

The probability of error is again determined by the "generalized E/N_o," now given by

$$\int_0^T r(t)\, q(t)\, dt = (r, K_1 r)_{T_1} = \underline{r} \cdot \underline{r} = \sum_{i=1}^{\infty} \lambda_i x_i^2 \quad .$$

Thus, since the input signal energy is $(x, x) = \underline{x} \cdot \underline{x}$, it follows that again $x(t) = \varphi_1(t)$ is the optimum signal to use to minimize probability of error for fixed input signal energy.

CHAPTER III
ERROR BOUNDS AND SIGNAL DESIGN FOR DIGITAL COMMUNICATION OVER FIXED TIME-CONTINUOUS CHANNELS WITH MEMORY

In Sec. I-E, a model was introduced which involves the transmission of one of M signals of T seconds duration through the channel of Fig. 1 and observation of the channel output for T_1 seconds. Subsequently, it was shown that a suitable matrix representation for this problem is

$$\underline{y} = [H]\,\underline{x} + \underline{n} \tag{22}$$

where $\underline{x}$, $\underline{y}$, and $\underline{n}$ are the (column) vectors representing the channel input, output, and additive noise, respectively, $[H]$ is a diagonal matrix, and $\underline{n}$ has statistically independent and identically distributed components. Using this representation and a slight generalization of recent results of Gallager,[74,75] this chapter will present the derivation of a bound on probability of error and provide some information on optimum signal design.[76,77]

A. VECTOR DIMENSIONALITY PROBLEM

Before proceeding with the derivation of an error bound, it is necessary to consider in detail the dimensionality of the vectors involved. In deriving the representation of Eq. (22), it was shown that the basis functions used in defining $\underline{x}$ are complete in the space of all $\mathcal{L}_2(0, T)$ signals, that is, in the space of all finite-energy signals defined on the interval $[0, T]$. Since it is well known that this space is infinite-dimensional,[64] it follows that, in general, the vectors, as well as the matrix, of Eq. (22) must be infinite-dimensional. In many cases, this infinite dimensionality is of no concern and mathematical operations can be performed in the usual manner. However, an attempt to define a "density function" for an infinite-dimensional random vector leads to conceptual as well as mathematical difficulties. Consequently, problems in which this situation arises are usually approached[54] by assuming initially that all vectors are finite-dimensional. The analysis is then performed and an attempt is made to show that a limiting form of the answer is obtained as the dimensionality becomes infinite. If such a limiting result exists, it is asserted to be the desired solution. This approach is used in the following derivations in which all vectors are initially assumed to be d-dimensional. However, for this problem, it will be shown that for minimum probability of error the signal vectors should be finite dimensional. Furthermore, it will be found that the "optimum" dimensionality is independent of d (assuming that d is large) and thus that the original restriction of finite d involves no loss of generality. This result is obtained because the λ_i in the $[H]$ matrix approach zero for large "i" and gives an indication of the useful "dimensionality" of the channel.

B. RANDOM CODING TECHNIQUE

This chapter is concerned with an investigation of the probability of error P_e for digital communication over the channel of Fig. 1. Ideally, the first step in this analysis would be the derivation of an expression for the probability of error under the assumption of an arbitrary set of M signals. This expression would then be minimized over all allowable signals to find both the minimum possible probability of error and the set of signals that achieve this. However, it will be found here, as is usually the case,[51,53] that an exact expression for P_e is impossible to evaluate by any means other than numerical techniques. Thus, it is necessary to

investigate some form of an approximate solution to this problem. The approach used here involves an application of the random coding technique to a suitable upper bound to P_e.

Conceptually, the random coding technique may be viewed in the following manner. First, an ensemble of codes† is constructed by selecting each code word of each code independently and at random according to a probability measure $p(\underline{x})$. In other words, for each code word

$$P_r\{\underline{x}: \xi_1 < x_1 < \xi_1 + d\xi_1, \xi_2 < x_2 < \xi_2 + d\xi_2, \ldots, \xi_k < x_k < \xi_k + d\xi_k, \ldots\}$$

$$= p(\underline{\xi})\, d\xi_1 d\xi_2 \ldots d\xi_k \ldots = p(\underline{\xi})\, d\underline{\xi}$$

and, furthermore, for any set of distinct code words, say $\underline{x}_1, \ldots, \underline{x}_s$,

$$p(\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_s) = p(\underline{x}_1)\, p(\underline{x}_2) \ldots p(\underline{x}_s) \quad .$$

Next, the probability of error for each code is evaluated. Finally, these values are averaged over the ensemble of codes to find an average probability of error $\overline{P}_e$. Viewed in this manner, it seems that the random coding technique only makes an impossible problem even more difficult. However, the simple expedient of reversing the order of ensemble averaging and evaluation of P_e, when coupled with a suitable bound on P_e, leads to a relatively simple expression for $\overline{P}_e$ for the channel of Fig. 1.

Accepting this statement, it is still not clear that knowledge of $\overline{P}_e$ is either meaningful or useful. For example, the knowledge of an average probability of error for an ensemble of codes is quite different from a knowledge of P_e for a single code. Furthermore, it is not a priori obvious that a knowledge of $\overline{P}_e$ will provide any information on the construction of good codes. Finally, it is not clear that the value of $\overline{P}_e$ will provide any indication of the minimum possible probability of error for a code since the ensemble averaging could include a large fraction of codes having a high probability of error.

Fortunately, these doubts can be resolved quite readily. The very fact that $\overline{P}_e$ is an average over an ensemble of codes implies that there must be at least a fraction $1/A$ ($A \geq 1$) of the codes in the ensemble that have a P_e less than $A\overline{P}_e$. Thus, for example, the construction of a code by choosing each code word independently and at random according to the probability measure $p(\underline{x})$ must lead, in 99 times out of 100, to a code having a P_e not greater than $100\,\overline{P}_e$. Although this procedure certainly does not yield a code having an absolute minimum probability of error, it is at present the only general approach known for constructing large codes that have a P_e that is even close to this minimum.‡

Therefore, the only problem that remains to establish the usefulness of the random coding technique is to determine the accuracy of $\overline{P}_e$ relative to the minimum possible P_e. For both discrete, memoryless channels and the time-discrete Gaussian channel, this problem has been studied by deriving a lower bound to P_e by means of the "sphere packing" argument.[51,53] For these channels it has been shown that the two exponents in the bounds to P_e and $\overline{P}_e$ are identical for all rates between a rate R_c and capacity. Furthermore, for the Gaussian channel the value of R_c is less than the rate at which a digital communication system with coding would be operated.

† For convenience, the terms "code" and "code word" are used here to mean, respectively, "a set of M finite energy signals of T seconds duration" and "one of a set of M finite energy signals of T seconds duration." In addition, no distinction is made between a signal $x_i(t)$ and the vector $\underline{x}_i$ representing this signal.

‡ The validity and importance of this approach have been demonstrated experimentally with the SECO machine.[50]

Thus, for this channel, the random coding technique provides a practical solution to the problem of determining probability of error.

Since the determination of a lower bound to P_e does not appear practical for the channel of Fig. 1, the approach used here has been to specialize the P_e expression to the case considered by Shannon[53] and then to compare this result to his. This analysis is performed in Sec. III-E and indicates that the bound obtained in this work is quite accurate at rates above R_c.

One point that has not been considered is the question of how the probability measure $p(\underline{x})$ should be selected. Aside from noting that the ensemble of codes should satisfy an energy constraint of the form of Eq. (1), the only statement that can be made is that a mathematically tractable function should be chosen which "seems reasonable" on the basis of experience and intuition. If the resulting expression for $\overline{P}_e$ can be shown to be sufficiently accurate, the problem is solved. Otherwise, another choice would be tried. As a practical matter, it is found that for the Gaussian channel of Fig. 1 the logical choice of a Gaussian p(x) of the form

$$p(\underline{x}) = \prod_i p_i(x_i)$$

where $p_i(x_i)$ is a one-dimensional Gaussian density function, leads to satisfactory results.

C. RANDOM CODING BOUND

This section will derive a random coding bound on probability of error for digital communication over the channel of Fig. 1. Let $\underline{x}_1, \ldots, \underline{x}_M$ be an arbitrary set of M d-dimensional code words for use with this channel and assume that the a priori probability for each code word is $1/M$. Let the probability density function for $\underline{y}$ given that $\underline{x}_j$ was transmitted be $p(\underline{y}|\underline{x}_j)$. Then, it is well known[51,54] that the detector which minimizes probability of error decides that $\underline{x}_j$ was transmitted if and only if

$$p(\underline{y}|\underline{x}_k) \leqslant p(\underline{y}|\underline{x}_j) \qquad \text{for all } k = 1, 2, \ldots, M \quad .$$

Thus, if a set of M characteristic functions[69] are defined as

$$\varphi_j(\underline{y}) \triangleq \begin{cases} 0 & [\underline{y}:p(\underline{y}|\underline{x}_k) \leqslant p(\underline{y}|\underline{x}_j) \quad \text{for all } k = 1, \ldots, M] \\ 1 & [\underline{y}:p(\underline{y}|\underline{x}_k) > p(\underline{y}|\underline{x}_j) \quad \text{for some } k = 1, \ldots, M] \end{cases}$$

it follows that the probability of error for this detector is given by

$$P_e = \frac{1}{M} \sum_{j=1}^{M} \int_{\underline{Y}} \varphi_j(\underline{y})\, p(\underline{y}|\underline{x}_j)\, d\underline{y} \triangleq \frac{1}{M} \sum_{j=1}^{M} P_j(e) \quad . \tag{23}$$

This expression, while valid for any M and any set of signals $\{s_j(t)\}$, is mathematically intractable for interesting values of M. Thus, it is necessary to derive a bound on P_e that is sufficiently accurate to be useful and yet is readily evaluated. The random coding technique discussed above when applied to a suitable upper bound to Eq. (23) gives such a result.

An obvious inequality is

$$\varphi_j(\underline{y}) \leq \left\{ \sum_{\substack{k=1\\k\neq j}}^{M} \left[\frac{p(\underline{y}|\underline{x}_k)}{p(\underline{y}|\underline{x}_j)}\right]^{1/1+\rho} \right\}^{\rho} \qquad \rho \geq 0$$

since the right-hand side is always non-negative and is not less than 1 when $p(\underline{y}|\underline{x}_j) < p(\underline{y}|\underline{x}_k)$ for some $j \neq k$. Thus,

$$P_j(e) \leq \int_{\underline{Y}} p(\underline{y}|\underline{x}_j)^{1/1+\rho} \left[\sum_{\substack{k=1\\k\neq j}}^{M} p(\underline{y}|\underline{x}_k)^{1/1+\rho} \right]^{\rho} d\underline{y} \quad . \tag{24}$$

Now, let each code word be chosen independently and at random according to a probability measure $p(\underline{x})$ and average both sides of Eq. (24) over this ensemble of codes. Then

$$\overline{P_j(e)} \triangleq \overline{P}_e \leq \int_{\underline{Y}} \overline{p(\underline{y}|\underline{x}_j)^{1/1+\rho}}\ \overline{\left[\sum_{\substack{k=1\\k\neq j}}^{M} p(\underline{y}|\underline{x}_k)^{1/1+\rho} \right]^{\rho}} d\underline{y} \tag{25}$$

where the bar denotes averaging with respect to the ensemble of codes and the independence of the selection of $\underline{x}_j$ and $\underline{x}_k$ has been used for the average under the integral. Equation (25) can be further upper bounded by noting that $\overline{z^{\rho}} \leq \bar{z}^{\rho}$ for $0 \leq \rho \leq 1$ (Ref. 78). Introducing this inequality into Eq. (25), and recalling that the average of a sum of random variables equals the sum of the individual averages, gives

$$\overline{P}_e < M^{\rho} \int_{\underline{Y}} \overline{p(\underline{y}|\underline{x})^{1/(1+\rho)}}^{(1+\rho)} d\underline{y} \qquad 0 \leq \rho \leq 1$$

or

$$\overline{P}_e < \exp\left[-TE(R,\rho)\right] \qquad 0 \leq \rho \leq 1 \tag{26}$$

where

$$E(R,\rho) = E_o(\rho) - \rho R$$

$$E_o(\rho) = -\frac{1}{T} \ln \int_{\underline{Y}} \left[\int_{\underline{X}} p(\underline{y}|\underline{x})^{1(1+\rho)} p(\underline{x})\, d\underline{x} \right]^{1+\rho} d\underline{y}$$

$$R = \frac{\ln M}{T} \quad .$$

This bound, which applies to any channel for which the indicated integrals exist, will now be specialized to the channel of Fig. 1. Recall that in deriving Eq. (26) it was assumed that the set of signals $\{s_j(t)\}$ were d-dimensional, i.e., each of the signals could be written as a linear combination of d basis functions. There was, however, no assumption made with respect to which set of d functions should be used. Thus, if I denotes the set of integers that specify the basis functions used and if the matrix representations of Theorems 1, 2, and 3 are recalled, it follows that for the channel of Fig. 1

$$p(\underline{y}|\underline{x}) = \prod_{i\in I} \frac{1}{\sqrt{2\pi}} \exp[-\tfrac{1}{2}(y_i - \sqrt{\lambda_i}\, x_i)^2] \quad . \tag{27}$$

Next, let the probability measure $p(\underline{x})$ which defines the random ensemble of codes be

$$p(\underline{x}) = \prod_{i\in I} (2\pi\sigma_i^2)^{-1/2} \exp[-\tfrac{1}{2}(x_i^2/\sigma_i^2)] \quad . \tag{28}$$

There are two reasons for this choice of $p(\underline{x})$:

(1) This form of p(x) results in a mathematically tractable expression for the error exponent of Eq. (26).

(2) When the resulting exponent is specialized to the time-discrete case considered by Shannon[53] it is within a few percent of his random coding exponent (see Sec. III-E). Furthermore, Shannon's exponent was shown to be identical to the exponent in a <u>lower</u> bound to probability of error over a range of rates that are of considerable practical interest.

Finally, assume an average power constraint on the ensemble of codes of the form

$$\sum_{i\in I} \sigma_i^2 = ST \quad . \tag{29}$$

Substituting Eqs. (27) and (28) into Eq. (26) gives, after evaluation of the integrals,

$$E(R, \rho, \underline{\sigma}) = \frac{\rho}{2T} \sum_{i\in I} \ln\left[1 + \frac{\lambda_i\sigma_i^2}{1+\rho}\right] - \rho R \tag{30}$$

where

$$\underline{\sigma} = (\sigma_1^2, \sigma_i^2, \ldots, \sigma_k^2, \ldots) \quad .$$

For fixed R, maximization of Eq. (30) over ρ, $\underline{\sigma}$, and the set I gives the desired random-coding error exponent. For convenience, let this maximization be performed in the order I, $\underline{\sigma}$, ρ.

Maximization over the set I is easily accomplished by recalling[64] that the λ_i are by assumption ordered so that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \ldots$. Thus, the monotonic property of $\ln x$ for $x \geq 1$ implies that $E(R, \rho, \underline{\sigma})$ is maximized over the set I by choosing $I = \{1, 2, \ldots, d\}$.

The maximization over $\underline{\sigma}$ is most readily accomplished by using the properties of convex functions[79] defined on a vector space. For this purpose, the following definitions and a theorem of Kuhn and Tucker[80] (in present notation) are presented:

(1) A region of vector space is defined as convex if for any two vectors $\underline{\alpha}$ and $\underline{\beta}$ in the region and for any λ, $0 \leq \lambda \leq 1$, the vector $\lambda\underline{\alpha} + (1-\lambda)\underline{\beta}$ is also in the region.

(2) A function $f(\underline{\alpha})$ whose domain is a convex region of vector space is defined as concave if, for any two vectors $\underline{\alpha}$ and $\underline{\beta}$ in the domain of f and for any λ, $0 < \lambda < 1$,

$$\lambda f(\underline{\alpha}) + (1-\lambda) f(\underline{\beta}) \leq f[\lambda\underline{\alpha} + (1-\lambda)\underline{\beta}] \quad .$$

From these definitions it follows that the region of Euclidean d-space defined by the vector $\underline{\sigma}$, with

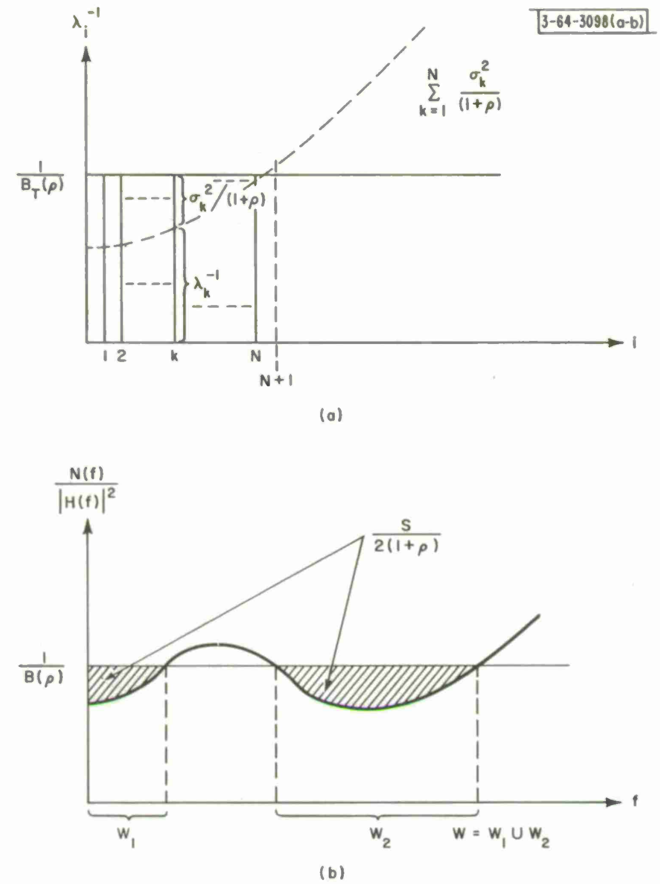

Fig. 5. Concerning interpretation of certain parameters in error exponent: (a) Interpretation of $B_T(\rho)$ and N; (b) Interpretation of $B(\rho)$ and W.

$$\sigma_i^2 \geqslant 0 \quad \text{and} \quad \sum_{i=1}^{d} \sigma_i^2 = TS$$

is a convex region of vector space, that $\ln x$ is a concave function for $x \geqslant 1$, that a sum of concave functions is concave, and thus that $E(R, \rho, \underline{\sigma})$ is a concave function of $\underline{\sigma}$.

Theorem 4 (Kuhn and Tucker).

Let $f(\underline{\sigma})$ be a continuous differentiable concave function in the region in which $\underline{\sigma}$ satisfies $\sum_{i=1}^{d} \sigma_i^2 = TS$ and $\sigma_i^2 > 0$, $i = 1, 2, \ldots, d$. Then a necessary and sufficient condition for $\underline{\alpha}$ to maximize f is

$$\left.\frac{\partial f(\underline{\sigma})}{\partial \sigma_i^2}\right|_{\underline{\sigma}=\underline{\alpha}} \leqslant A \qquad \text{for all } i \text{ with equality if and only if } \alpha_i \neq 0$$

where A is a constant independent of i whose value is adjusted to satisfy the constraint $\sum_{i=1}^{d} \sigma_i^2 = TS$.

It follows that the $\underline{\sigma}$ maximizing Eq. (30) must satisfy

$$\frac{\partial E(R, \rho, \underline{\sigma})}{\partial \sigma_i^2} = \frac{\rho}{2T} \frac{\lambda_i/(1+\rho)}{1 + (\lambda_i \sigma_i^2)/(1+\rho)} \leqslant A \qquad \text{all } i = 1, \ldots, d$$

with equality if and only if $\sigma_i^2 > 0$. Thus,

$$\sigma_i^2 = \begin{cases} (1+\rho)\left[\dfrac{1}{B_T(\rho)} - \dfrac{1}{\lambda_i}\right] & i = 1, 2, \ldots, N \\ 0 & i = N+1, \ldots, d \end{cases} \tag{31}$$

where N is defined by

$$\lambda_N > B_T(\rho) \geqslant \lambda_{N+1}$$

and

$$\frac{1}{B_T(\rho)} \triangleq \frac{\rho}{2TA(1+\rho)} .$$

The value of $B_T(\rho)$, and thus N, is chosen to satisfy the constraint

$$\sum_{i=1}^{N} \sigma_i^2 = TS$$

which yields

$$\frac{1}{B_T(\rho)} = \frac{\dfrac{ST}{1+\rho} + \sum_{i=1}^{N} \lambda_i^{-1}}{N} . \tag{32}$$

A convenient method for interpreting $B_T(\rho)$ and N is presented in Fig. 5(a). This is simply the discrete form of the well-known water-pouring interpretation discussed by Fano[51] and others for the special case of channel capacity. Substituting Eqs. (31) and (32) into Eq. (30) gives

$$E(R,\rho) = \frac{\rho}{2T} \sum_{i=1}^{N} \ln \frac{\lambda_i}{B_T(\rho)} - \rho R \quad . \tag{33}$$

Maximization over ρ is accomplished using standard techniques of differential calculus and leads to the final result

$$E_T(\rho) = \left(\frac{\rho}{1+\rho}\right)^2 \frac{S}{2} B_T(\rho) \qquad R(1) \leqslant R \leqslant R(0) \tag{34}$$

where

$$R(\rho) = \frac{1}{2T} \sum_{i=1}^{N} \ln \frac{\lambda_i}{B_T(\rho)} - \frac{\rho}{(1+\rho)^2} \frac{S}{2} B_T(\rho) \qquad 0 \leqslant \rho \leqslant 1 \tag{35}$$

and

$$E_T(R) = \frac{1}{2T} \sum_{i=1}^{N} \ln \frac{\lambda_i}{B_T(1)} - R \qquad 0 \leqslant R \leqslant R(1) \quad . \tag{36}$$

A bound that is in some cases more useful, and in all cases more readily evaluated, can be derived by considering Eqs. (34) to (36) for $T \to \infty$. It is shown in Appendix B that the resulting form for the exponent is

$$E(\rho) = \left(\frac{\rho}{1+\rho}\right)^2 \frac{S}{2} B(\rho) \qquad R_c \leqslant R \leqslant C \tag{37}$$

$$R(\rho) = \int_W \ln \frac{|H(f)|^2}{N(f)\, B(\rho)}\, df - \frac{\rho}{(1+\rho)^2} \frac{S}{2} B(\rho) \qquad 0 \leqslant \rho \leqslant 1 \tag{38}$$

$$E(R) = \int_W \ln \frac{|H(f)|^2}{N(f)\, B(1)}\, df - R \qquad 0 \leqslant R \leqslant R_c \tag{39}$$

where

$$C = R(0)$$

$$R_c = R(1)$$

$$\frac{1}{B(\rho)} = \frac{\dfrac{S}{2(1+\rho)} + \displaystyle\int_W \frac{N(f)}{|H(f)|^2}\, df}{W}$$

$$W = \left[+f: \frac{|H(f)|^2}{N(f)} \geqslant B(\rho)\right] \quad .$$

A convenient method for interpreting the significance of $B(\rho)$ and W is illustrated in Fig. 5(b). Pertinent properties of the exponent of Eqs. (37) to (39) are presented in Fig. 6.

D. BOUND FOR "VERY NOISY" CHANNELS

In this section, an asymptotic form for the bound of Eqs. (37) to (39) is derived for the condition $S \to 0$. Consider first the bound for $0 \leqslant \rho \leqslant 1$ and recall (Fig. 1) that N(f) is assumed to be normalized so that

$$\max_f \frac{|H(f)|^2}{N(f)} = 1 \quad .$$

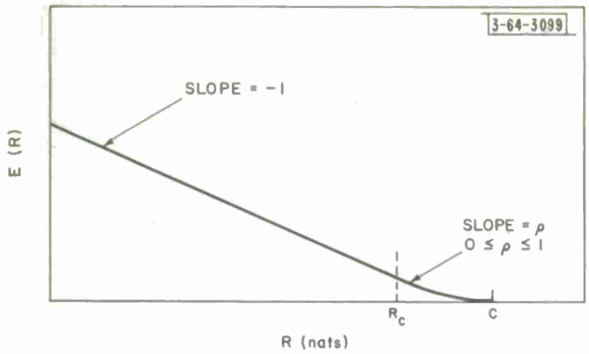

Fig. 6. Random coding exponent for channel in Fig. 1.

Thus, for S "sufficiently small," it is clear from the water-pouring interpretation of Fig. 5(b) that except for pathological H(f) and N(f),†

$$\frac{|H(f)|^2}{N(f)} \approx 1 \qquad \text{for } f \in W \quad .$$

Introducing this approximation into the expression for $B(\rho)$ gives

$$B(\rho) \approx \frac{1}{1 + \frac{S}{2W(1+\rho)}} \approx 1 - \frac{S}{2W(1+\rho)} \qquad \text{for } S \to 0 \quad .$$

In view of this, Eq. (37) becomes to first order in S

$$E(\rho) \approx \left(\frac{\rho}{1+\rho}\right)^2 \frac{S}{2} \qquad R_c \leqslant R \leqslant C \quad . \tag{40}$$

Introducing the same approximations into Eq. (38) gives

$$R(\rho) \approx \int_W \ln \frac{1}{B(\rho)}\, df - \frac{\rho}{(1+\rho)^2} \frac{S}{2}$$

$$\approx \int_W \left[\frac{1}{B(\rho)} - 1\right] df - \frac{\rho}{(1+\rho)^2} \frac{S}{2}$$

$$\approx \frac{S}{2(1+\rho)} - \frac{\rho}{(1+\rho)^2} \frac{S}{2} = \frac{S}{2(1+\rho)^2} \qquad 0 \leqslant \rho \leqslant 1 \quad . \tag{41}$$

Finally, elimination of ρ between Eqs. (40) and (41) gives

$$E(R) \approx C \left[1 - \sqrt{\frac{R}{C}}\right]^2 \qquad R_c \leqslant R \leqslant C \tag{42}$$

where

$$C = \frac{S}{2}$$

$$R_c = \frac{S}{8} = \frac{C}{4} \quad .$$

In a similar manner, it is found that Eq. (39) becomes approximately

$$E(R) \approx C \left[\frac{1}{2} - \frac{R}{C}\right] \qquad 0 \leqslant R \leqslant R_c \quad . \tag{43}$$

†Note that due to the normalization of N(f) the statement that S is "sufficiently small" is equivalent to the statement that a suitably defined signal-to-noise ratio is "sufficiently small."

The result of Eqs. (42) and (43) is of interest for several reasons.

(1) It is independent of the filter and noise spectral characteristics and is thus a "universal" bound for the channel of Fig. 1.

(2) It is identical to the exponent in the bound on probability of error for the transmission of orthogonal signals over a white Gaussian channel.[51] This implies that for "sufficiently small" signal power the memory of the channel in Fig. 1 has negligible effect on probability of error.

(3) It agrees precisely with the small signal-to-noise ratio (SNR) bound found by Shannon for the band-limited channel,[53] and is identical except for the definition of C, to a bound found by Gallager[75] for "very noisy" discrete memoryless channels. Thus, the bound of Eqs. (42) and (43) can, when C is appropriately defined, be considered to be a "universal" bound for "very noisy" time-invariant channels.

E. IMPROVED LOW-RATE RANDOM CODING BOUND

The previous sections have presented a random coding bound on probability of error for the channel of Fig. 1. As is usually the case with random coding bounds,[51,53] this bound can be shown to be quite poor under the conditions of low rate and high signal power.† In fact, it is not difficult to show that the true exponent in the bound on the smallest attainable probability of error differs from the random coding exponent by an arbitrarily large amount as the rate approaches zero and the signal power approaches infinity. This section presents an improved low-rate random coding bound based upon a slight generalization of recent work by Gallager[75] that overcomes this difficulty.

Before proceeding with the derivation of an improved bound, it is important to consider briefly the reason for the inaccuracy of the random coding bound. As indicated previously, the random coding bound is, in principle, obtained in the following manner. First, an ensemble of codes is constructed by selecting each code word of each code independently and at random according to a probability measure $p(\underline{x})$. Next, the probability of error is calculated for each code. Finally, these values of P_e are averaged to obtain $\overline{P}_e$. Note, however, that nothing in this procedure precludes the possibility that a small fraction of the codes in the ensemble may have a P_e considerably greater than that for the remaining codes. Thus, it is possible that $\overline{P}_e$ could be determined almost entirely by a small percentage of high P_e codes. (This is simply illustrated by considering a hypothetical situation in which 1 percent of the codes have a P_e of 10^{-1} while the remaining 99 percent have a P_e of 10^{-10}.) An improved bound is derived here by expurgating the high probability of error code words from each code in the ensemble used previously.

Consider the channel of Fig. 1 and let $\underline{x}_1, \underline{x}_2, \dots, \underline{x}_M$ be a set of code words for use with this channel. Then, given that $\underline{x}_j$ is transmitted, it follows from Eq. (24) that

$$P_j(e) \leq \int_{\underline{Y}} p(\underline{y}|\underline{x}_j)^{1/2} \left[\sum_{\substack{k=1 \\ k \neq j}}^{M} p(\underline{y}|\underline{x}_k)^{1/2} \right] d\underline{y} \tag{44}$$

† Comments analogous to those in the immediately preceding footnote also apply to the term "high signal power."

or equivalently

$$P_j(e) \leqslant \sum_{\substack{k=1 \\ k \neq j}}^{M} Q(\underline{x}_j, \underline{x}_k) \tag{45}$$

where

$$Q(\underline{x}_j, \underline{x}_k) = \int_{\underline{Y}} p(\underline{y}|\underline{x}_j)^{1/2}\, p(\underline{y}|\underline{x}_k)^{1/2}\, d\underline{y} \quad .$$

From these equations it is clear that $P_j(e)$ is a function of each of the code words in the code. Thus, for a random ensemble of codes in which each code word is chosen with a probability measure $p(\underline{x})$, $P_j(e)$ will be a random variable, and it will be meaningful to discuss the probability that $P_j(e)$ is not less than some number A, that is, $\Pr\{P_j(e) \geqslant A\}$. To proceed further, a simple bound on this probability is required. Following Gallager,[75] let a function $\varphi_j(\underline{x}_1, \ldots, \underline{x}_M)$ be defined as

$$\varphi_j(\underline{x}_1, \ldots, \underline{x}_M) = \begin{cases} 1 & \text{if } P_j(e) \geqslant A \\ 0 & \text{if } P_j(e) < A \end{cases} \quad . \tag{46}$$

Then, with a bar used to indicate an average over the ensemble of codes, it follows directly that

$$\Pr\{P_j(e) \geqslant A\} = \overline{\varphi_j(\underline{x}_1, \ldots, \underline{x}_M)} \quad . \tag{47}$$

From Eq. (45) an obvious inequality is

$$\varphi_j(\underline{x}_1, \ldots, \underline{x}_M) \leqslant A^{-s} \sum_{\substack{k=1 \\ k \neq j}}^{M} Q(\underline{x}_j, \underline{x}_k)^s \qquad 0 < s \leqslant 1 \tag{48}$$

since the right-hand side is always non-negative (for $A > 0$) and is not less than 1 when $P_j(e) \geqslant A$ and $0 < s \leqslant 1$. Thus,

$$\Pr\{P_j(e) \geqslant A\} \leqslant A^{-s} \sum_{\substack{k=1 \\ k \neq j}}^{M} \overline{Q(\underline{x}_j, \underline{x}_k)^s} \qquad 0 < s \leqslant 1 \tag{49}$$

where, due to the statistical independence of $\underline{x}_j$ and $\underline{x}_k$ over the ensemble of codes,

$$\overline{Q(\underline{x}_j, \underline{x}_k)^s} = \int_{\underline{X}} \int_{\underline{X}} \left[\int_{\underline{Y}} p(\underline{y}|\underline{x}_j)^{1/2}\, p(\underline{y}|\underline{x}_k)^{1/2}\, d\underline{y} \right]^s p(\underline{x}_j)\, p(\underline{x}_k)\, d\underline{x}_j\, d\underline{x}_k \quad .$$

In this form it is clear that $\overline{Q(\underline{x}_j, \underline{x}_k)^s}$ is independent of j and k and therefore that Eq. (49) reduces to

$$\Pr\{P_j(e) \geqslant A\} \leqslant (M-1)\, A^{-s}\, \overline{Q(\underline{x}_j, \underline{x}_k)^s} \qquad 0 < s \leqslant 1 \quad . \tag{50}$$

At this point it is convenient to choose A to make the right-hand side of Eq. (50) equal to 1/2. Solving for the value of A gives

$$A = [2(M-1)]^{\rho} \left[\overline{Q(\underline{x}_j, \underline{x}_k)^{1/\rho}}\right]^{\rho} \qquad \rho \geqslant 1 \tag{51}$$

where $\rho \triangleq 1/s$. Now, let all code words in the ensemble for which $P_j(e) \geqslant A$ be expurgated. Then from Eq. (51) all remaining code words satisfy

$$P_j(e) < [2M]^{\rho} \left[\overline{Q(\underline{x}_j, \underline{x}_k)^{1/\rho}}\right]^{\rho} \qquad \rho \geqslant 1 \quad . \tag{52}$$

Furthermore, since $\Pr\{P_j(e) \geqslant A\} \leqslant 1/2$, it follows that the average number of code words M' remaining in each code satisfies $M' \geqslant M/2$. Thus, there exists at least one code in the expurgated set containing not less than $M/2$ code words and having a probability of error for each code word satisfying Eq. (52). By setting $\exp[RT] = M/2$, it follows that there exists a code for which

$$P_e < 4^{\rho} \exp[-TE^{e}(R, \rho)] \qquad \rho \geqslant 1 \tag{53}$$

where

$$E_o^e(R, \rho) = E_o^e(\rho) - \rho R$$

and

$$E_o^e(\rho) \triangleq \frac{-\rho}{T} \ln \overline{Q(\underline{x}_j, \underline{x}_k)^{1/\rho}} \quad .$$

This bound will now be applied to the channel of Fig. 1. As before, let

$$p(\underline{y}|\underline{x}) = \prod_{i \in I} \frac{1}{\sqrt{2\pi}} \exp[-\tfrac{1}{2}(y_i - \sqrt{\lambda_i}x_i)^2] \tag{54}$$

let $p(\underline{x})$ be chosen as

$$p(\underline{x}) = \prod_{i \in I} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x_i}{\sigma_i}\right)^2\right] \tag{55}$$

and let the input power constraint be given by

$$\sum_{i \in I} \sigma_i^2 = ST \quad . \tag{56}$$

[This form for $p(\underline{x})$ has been chosen primarily for mathematical expediency. However, results obtained by Gallager[75] indicate that it is indeed a meaningful choice from the standpoint of maximizing the resulting exponent.] Substituting Eqs. (55) and (56) in Eq. (53) yields, after the integrals are evaluated,

$$E_o^2(\rho, \underline{\sigma}) = \frac{\rho}{2T} \sum_{i \in I} \ln\left[1 + \frac{\lambda_i \sigma_i^2}{2\rho}\right] \qquad \rho \geqslant 1 \tag{57}$$

where

$$\underline{\sigma} \triangleq (\sigma_1^2, \ldots, \sigma_j^2, \ldots, \sigma_\ell^2, \ldots) \quad .$$

For fixed R and T, maximization of $E_o^2(\rho) - \rho R$ over ρ, $\underline{\sigma}$, and the set I gives the desired bound. [In the maximization over ρ it is assumed that $RT \gg \ln 4$. This allows the factor of 4^ρ in Eq. (53) to be neglected in performing the maximization and involves no loss of generality since $RT = \ln M$ and large values of M are of interest.] Comparison of Eqs. (53) and (57) with Eq. (30) reveals a strong similarity in the analytical expressions for the two bounds. As a result, the maximization procedure used previously can be applied without change to this problem, yielding the final result

$$P_e < 4^\rho \exp[-TE_T^e(\rho)] \qquad \rho \geqslant 1 \quad . \tag{58}$$

where

$$E_T^e(\rho) \triangleq \frac{S}{4} B_T(2\rho - 1)$$

$$R(\rho) \triangleq \frac{1}{2T} \sum_{i=1}^{N} \ln \frac{\lambda_i}{B_T(2\rho - 1)} - \frac{S}{4} \frac{B_T(2\rho - 1)}{\rho}$$

$$\frac{1}{B_T(\cdot)} \triangleq \frac{\dfrac{ST}{[1 + (\cdot)]} + \sum_{i=1}^{N} \lambda_i^{-1}}{N}$$

and N satisfies

$$\lambda_N > B_T(2\rho - 1) \geqslant \lambda_{N+1} \quad .$$

As before, a bound that is more readily evaluated can be derived by considering Eq. (58) for $T \to \infty$. The result, which is proved in Appendix B, is

$$P_e < 4^\rho \exp[-TE^e(\rho)] \qquad \rho \geqslant 1 \tag{59}$$

where

$$E^e(\rho) \triangleq \frac{S}{4} B(2\rho - 1)$$

$$R(\rho) \triangleq \int_W \ln \frac{|H(f)|^2}{N(f)\, B(2\rho - 1)}\, df - \frac{S}{4} \frac{B(2\rho - 1)}{\rho}$$

$$\frac{1}{B(\cdot)} \triangleq \frac{\dfrac{S}{2[1 + (\cdot)]} + \int_W \dfrac{N(f)}{|H(f)|^2}\, df}{W}$$

$$W \triangleq \left\{+f: \frac{|H(f)|^2}{N(f)} \geqslant B(2\rho - 1)\right\} \quad .$$

Figure 7 presents the pertinent characteristics of this bound and relates it to the previous random coding bound. Note from Eq. (59) that $B(2\rho - 1)$ and W can be interpreted in terms of the water-pouring picture of Fig. 5(b) by simply replacing $S/[2(1 + \rho)]$, where $0 \leqslant \rho \leqslant 1$ with $S/4\rho$, where $\rho \geqslant 1$.

This completes the derivation of the error bounds for the channel of Fig. 1. There remains the problem of investigating the accuracy of these results since there is no assurance that they

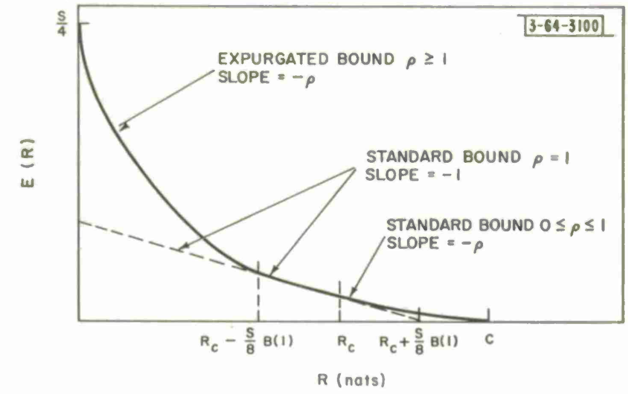

Fig. 7. Error exponent for channel in Fig. 1.

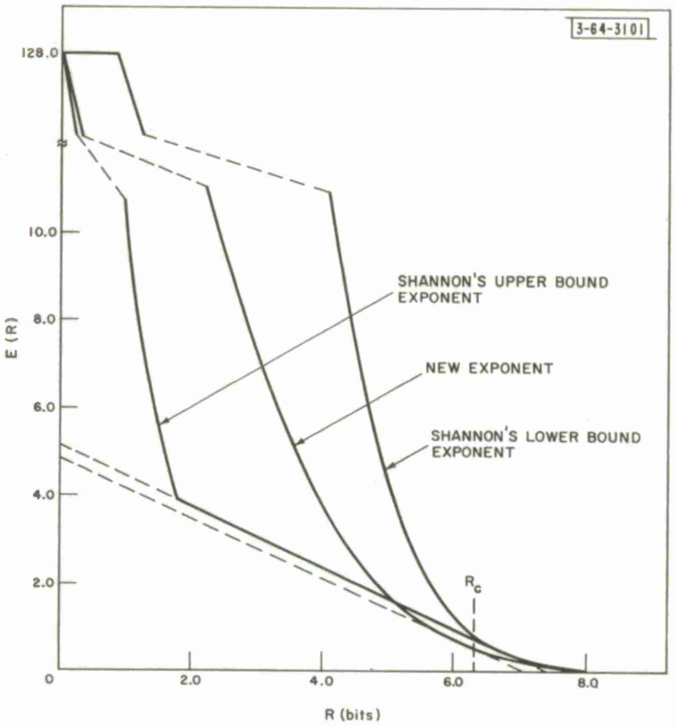

Fig. 8. Comparison of error exponents.

will be sufficiently accurate to be useful.† As noted in Sec. III-B, the only practical approach to this problem is to specialize the bounds obtained here to the case considered by Shannon[53] and then compare the two results. This is most readily accomplished by considering the present bounds in the form of Eqs. (34) to (36) and Eq. (58) for the case

$$\lambda_i = \begin{cases} 1 & i = 1, 2, \dots, 2T \quad \text{(T an integer)} \\ 0 & i > 2T \quad . \end{cases}$$

When evaluated these become

$$E(\rho) = \left(\frac{\rho}{1+\rho}\right)^2 \frac{S}{2} \left[1 + \frac{S}{2(1+\rho)}\right]^{-1} \qquad R_c \leqslant R \leqslant C$$

$$R(\rho) = \ln\left[1 + \frac{S}{2(1+\rho)}\right] - \frac{E(\rho)}{\rho} \text{ nats/second} \qquad 0 \leqslant \rho \leqslant 1$$

$$E(R) = \ln\left[1 + \frac{S}{4}\right] - R \qquad 0 \leqslant R \leqslant R_c$$

and

$$E^e(\rho) = \frac{S}{4}\left[1 + \frac{S}{4\rho}\right]^{-1} \qquad \rho \geqslant 1$$

$$R(\rho) = \ln\left[1 + \frac{S}{4\rho}\right] - \frac{E^e(\rho)}{\rho} \text{ nats/second} \qquad \rho \geqslant 1$$

and are plotted in Fig. 8 for $S/2 = 256$. The corresponding bounds of Shannon are also plotted in Fig. 8 and have been obtained by observing that the relation between the present notation and that of Shannon is

$$2T = n$$

$$S/2 = A^2$$

$$R = 2R_s \quad .$$

It should be noted that the new bound is tighter than Shannon's for low rates although somewhat weaker for rates near capacity. Furthermore, the new bound is quite close to Shannon's <u>lower</u> bound exponent over a range of rates around R_c that are of considerable practical interest.

In order to obtain additional insight into the form of the error bounds for different channels, it is of interest to evaluate the bounds for the channel defined by

$$H(f) = \frac{1}{1 + jf}$$

$$N(f) = 1 \quad .$$

Substitution of these expressions into Eqs. (37) to (39) and Eq. (59) gives

$$E(\rho) = \left(\frac{\rho}{1+\rho}\right)^2 \frac{S}{2} \left\{1 + \left[\frac{3S}{4(1+\rho)}\right]^{2/3}\right\}^{-1} \qquad R_c \leqslant R \leqslant C$$

$$R(\rho) = 2\left\{\left[\frac{3S}{4(1+\rho)}\right]^{1/3} - \tan^{-1}\left[\frac{3S}{4(1+\rho)}\right]^{1/3}\right\} - \frac{E(\rho)}{\rho} \text{ nats/second} \qquad 0 \leqslant \rho \leqslant 1$$

$$E(R) = 2\left[\left(\frac{3S}{8}\right)^{1/3} - \tan^{-1}\left(\frac{3S}{8}\right)^{1/3}\right] - R \qquad 0 \leqslant R \leqslant R_c$$

† See Sec. III-B for a discussion of this problem and of the procedure normally used to investigate such bounds.

and

$$E^e(\rho) = \frac{S}{4}\,[1 + (\frac{3S}{8\rho})^{2/3}]^{-1} \qquad\qquad \rho \geqslant 1$$

$$R(\rho) = 2\,[(\frac{3S}{8\rho})^{1/3} - \tan^{-1}(\frac{3S}{8\rho})^{1/3}] - \frac{E^e(\rho)}{\rho} \quad \text{nats/second}$$

Figure 9 presents these bounds for $S = 10^3$. It is interesting to note that the only basic difference between the curves in Figs. 8 and 9 is that the ratio R_c/C is significantly smaller for the latter curve.

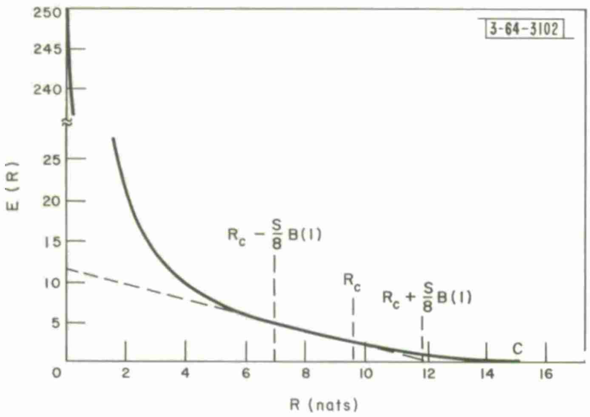

Fig. 9. Error exponents for channel in Fig. 1 with $H(f) = [1 + jf]^{-1}$, $N(f) = 1$, and $S = 10^3$.

A fact of considerable practical importance can be determined from the above examples. At present, the highest rate at which practical coding devices can operate is R_{comp} (Ref. 82) – the rate axis intercept of the straight-line portion of the unexpurgated random coding bound.† For "noisy" channels [see Eq. (43)], R_{comp} is readily found to be $C/2$; a somewhat disappointing result in view of a natural desire to signal at rates approaching C. However, Figs. 8 and 9 demonstrate that the situation is quite different for moderately large signal powers. For the band-limited channel, R_{comp} is essentially 1 bit/second less than capacity for all $S \gtrsim 10^2$, and in fact $R_{comp}/C \to 1$ for $S \to \infty$. For the single-pole channel, it is readily shown that $R_{comp} \to 0.8\,C$ for $S \to \infty$ with this relation being quite accurate for $S \gtrsim 10^3$. Thus, at high signal powers it is possible to achieve data rates quite close to capacity using existing coding and decoding techniques.

In concluding this section, one final point should be made concerning the error bounds. As indicated above, practical coding devices operate at rates less than R_{comp}. However, the use of such devices would normally imply a desire to use the channel as efficiently as possible, i.e., to use a rate near R_{comp}. Thus, the fact that the unexpurgated random coding bound applies for all rates above $R_c - (S/8)\,B(1)$ coupled with the fact that $R_{comp} > R_c$ implies that there would seldom be any practical interest in the expurgated bound. Furthermore, it is clear from Fig. 7 that R_c and C are the crucial factors in determining the unexpurgated bound;‡ in other words, given R_c and C, an exponent that is sufficiently accurate for engineering purposes can be obtained graphically by simply using a French curve to draw the exponent between R_c and C.

† Note from Eqs. (38) and (39) that for any channel and for any signal power S, $R_{comp}(S) = C(S/2)$, i.e., the expression for R_{comp} is identical to that for C with S replaced by S/2.

‡ This, of course, assumes that C is known as a function of S so that R_{comp} can be determined from the relation $R_{comp}(S) = C(S/2)$.

F. OPTIMUM SIGNAL DESIGN IMPLICATIONS OF CODING BOUNDS

This section is concerned with presenting an answer to the third question of Sec. I-E; namely, how should the signals that are to be transmitted through the channel of Fig. 1 be constructed so as to minimize probability of error. The need for such an answer is made clear when the wide variety of techniques (such as AM/DSB, AM/VSB, PM, differential PM, FM, etc.) presently used for telephone line data transmission are considered.

Initially, it might appear somewhat surprising to expect any information concerning signal design from the error bounds derived above. For example, the derivation of error bounds for the white Gaussian channel[51,53] provides no insight into "good" signaling waveforms and in fact it makes no difference – the signals are simply linear combinations of <u>any</u> set of orthonormal functions.† However, for the channel of Fig. 1, the previous analysis shows that "good" signals are <u>finite</u> linear combinations of a <u>particular</u> set of orthonormal functions, namely, those of Theorems 1 to 3. Furthermore, the use of any other set of orthonormal functions would, in general, lead to "good" signals that were <u>infinite</u> linear combinations of these functions. Thus, in this case, the error bounds do indeed provide significant insight into how "good" signals should be constructed.

To obtain an understanding of how the error bounds provide an answer to the signal design question, it is necessary to reconsider briefly the original formulation of the problem. As indicated in Sec. I-E, it was desired to formulate the problem of digital communication over fixed time-continuous channels with memory in such a way that the subsequent analysis would lead to a bound on the best possible performance. Thus, no practical restrictions were introduced with respect to the form of the signals; it was simply stated that the channel would be used <u>once</u> for T seconds to transmit a signal of T seconds duration. Since T was completely arbitrary, this appeared to be the most general statement possible.‡ Following this formulation of the problem, basis functions were found for use in the vector space representation of the signals. Since the basis functions were shown to be complete, this representation introduced no restrictions on the form of possible "good" signals. Next, the vector space signal representation was used with the random coding technique to obtain an upper bound to probability of error for a random ensemble of codes. As noted in Sec. III-A, this derivation initially restricted the signals to be d-dimensional where d was finite but could be taken to be arbitrarily large. However, the result of optimizing the random coding bound over the structure of the ensemble of codes showed that only a lesser number N of the input signal coordinates should be used. [See Eq. (31) and the preceding maximization over $\underline{\sigma}$.] Thus, allowing d to become infinite after the optimization procedure removes the initial finite dimensionality restriction. This leads to the conclusion that of all possible structures for a random ensemble of codes, the best is the one in which each code word in the ensemble is a finite linear combination of the first N eigenfunctions. Furthermore, this result demonstrates that "optimum" digital communication over the channel of Fig. 1 involves use of the channel only <u>once</u> to transmit one of these signals.

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† Clearly, practical considerations (e.g., a peak-power limitation) might make one set of functions more desirable than another.

‡ For example, this statement does not preclude the possibility that "good" signals might be linear combinations of time translates of a relatively short and simple signal, it just does not assume this.

In conclusion, two points should be made concerning this "optimum" signal structure.

(1) In retrospect, it is almost obvious that good signals should be of this form. As noted before, the eigenvalues λ_i are effectively energy transfer ratios. Furthermore, when the eigenvalues are ordered so that $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \ldots$, it is known[64] that $\lambda_i \to 0$ for $i \to \infty$. Thus, since the input signals have an energy constraint and since the channel noise is Gaussian, it would intuitively seem most unwise to place a large amount of signal energy on an eigenfunction that was severely "attenuated" in passing through the filter. Clearly, the theoretical analysis confirms this reasoning and in addition, provides an explicit method for determining the number of eigenfunctions N that should be used.

(2) Although the previous discussion has demonstrated the optimality of this signal structure from a theoretical viewpoint, it is clear that from an engineering standpoint it is not practical; i.e., the generation of a large number of eigenfunctions lasting for days or weeks is simply not feasible. Thus, it becomes important to investigate other more practical forms of basis functions in an attempt to find signals that can be readily generated in practice and are also nearly as "good" as the optimum signals. This problem is considered in Chapter IV.

G. DIMENSIONALITY OF COMMUNICATION CHANNEL

The dimensionality of a channel is a concept of interest to communication engineers. This concept is frequently discussed in terms of the noiseless, band-limited channel defined by

$$H(f) = \begin{cases} 1 & |f| \leqslant W \\ 0 & \text{elsewhere} \end{cases}$$

$$N(f) = 0 \quad .$$

For this channel, the dimensionality is usually considered to be the number of linearly independent signals that can be transmitted in T seconds and recovered without mutual interference. By an argument based on the sampling representation for band-limited signals it is concluded that the channel dimensionality is 2TW, since use of $(\sin t)/t$ signals allows transmission and recovery of 2W independent signals per second. There are, however, several fundamental criticisms of this approach.

(1) Since $(\sin t)/t$ signals are not time-limited, the statement that 2TW signals can be transmitted in T seconds involves an (arbitrary) approximation.

(2) It is not clear how this approach should be used to define the dimensionality of a band-limited but nonrectangular channel or of a non-band-limited channel. For example, if $H(f) = [1 + j\omega]^{-1}$, how is W to be defined? Conversely, if the channel is band-limited but nonrectangular, its impulse response is, in general,

$$h(t) = \sum_i h_i \frac{\sin \pi(2TW - i)}{\pi(2TW - i)} \quad .$$

Thus, transmission of $(\sin t)/t$ signals leads to received signals having mutual (or intersymbol) interference.

(3) In view of Theorem 1, an alternate and considerably more general definition of channel dimensionality is simply the number of orthonormal signals of T seconds duration that can be obtained which remain orthogonal over some interval at the channel output. This definition is more appealing, since it applies to any channel and since the intersymbol interference is zero between all output signals. However, it was shown in Theorem 1

> that the set of $\varphi_i(t)$ having this property are complete† and therefore infinite in number. Thus, in contrast to the finite dimensionality of the original approach, this definition indicates that all channels are infinite dimensional. Clearly, this represents no improvement over the previous definition.

Before presenting a definition of dimensionality that overcomes these problems, it is important to consider in greater detail the infinite dimensional result just obtained. As indicated previously, the eigenvalues λ_i corresponding to each $\varphi_i(t)$, are the energy "transfer ratio" of the filter for that eigenfunction, i.e.,

$$\lambda_i = \frac{(h\varphi_i, h\varphi_i)_{T_1}}{(\varphi_i, \varphi_i)} = \frac{\int_0^{T_1} \left[\int_0^T \varphi_i(\tau)\, h(t-\tau)\, d\tau\right]^2 dt}{\int_0^T \varphi_i^2(t)\, dt} \quad .$$

Furthermore, when the λ_i are ordered so that $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \ldots$, it is known[64] that $\lambda_i \to 0$ for $i \to \infty$. Assuming normalized $\varphi_i(t)$, this implies that when $\varphi_i(t)$ is transmitted the output signal energy approaches zero for large "i." Thus, since any physical situation involves measurement inaccuracies (noise), it is intuitively clear that the useful dimensionality of a channel is indeed finite, i.e., all the $\varphi_i(t)$ whose eigenvalues are "too small" are unimportant in determining the channel dimensionality.

From this discussion it is clear that a useful definition of the dimensionality of a communication channel must include a meaningful definition of "too small." The following definition of dimensionality which is based on the optimum signal results of Sec. III-F satisfies this requirement.

> Let $\{x_j(t)\}$ be a set of signals for use with the channel of Fig. 1 and let the $\{x_j(t)\}$ be selected in the optimum manner of Sec. III-F, i.e., each $x_j(t)$ is of the form
>
> $$x_j(t) = \sum_{i=1}^{N} x_{ij}\varphi_i(t) \qquad 0 \leqslant t \leqslant T$$
>
> where N is determined by Eq. (31). Then the <u>dimensionality</u> D of the channel is defined as‡
>
> $$D = \lim_{T\to\infty} \left[\frac{N}{T}\bigg|_{\rho=0}\right] \quad .$$

Note that this is effectively a "dimensionality per second" definition as opposed to the "dimensionality per T seconds" definition considered previously. This normalized definition is used so that a finite number will be obtained for the dimensionality in the limit $T \to \infty$.

In concluding this section, several points concerning this definition should be mentioned.

(1) From Appendix B it follows that D = 2W, where W is defined by Eq. (39) and the water pouring interpretation of Fig. 5(b).

---

† Although the completeness proof for Theorem 1 does not apply to the rectangular band-limited channel $\left(\int_{-\infty}^{\infty} |h(t)|\,dt = \infty\right)$ it is possible to show that the $\phi_i(t)$ are also complete for this case.

‡ Recall that N is the dimensionality (Sec. II-C) of the set of transmitted signals and that for $T \to \infty$ and $\rho \to 0$, $R \to C$. Thus, the channel dimensionality is defined as the limiting (normalized) dimensionality of the set of signals that achieve a rate arbitrarily close to capacity.

(2) The presence of noise is considered simply and directly in the evaluation of W and thus of D.

(3) It is satisfying to note that application of this definition to the band-limited channel defined by

$$H(f) = \begin{cases} 1 & |f| \leqslant W \\ 0 & \text{elsewhere} \end{cases}$$

$$N(f) = N_o > 0$$

gives D = 2W.

(4) This definition applies to any channel whether or not it is band-limited and flat.

(5) Calculations indicate that $N \simeq TD = 2TW$ [where N is defined by Eq. (31) and W is defined as in (1)] when T is only moderately large. For example, Slepian[73] has shown that, for the band-limited channel defined in (3), the error in this relation $(N - 2TW)$ is not greater than unity for $2TW \gtrsim 2$. Furthermore, at the other extreme calculations by the author for the channel

$$H(f) = (1 + jf)^{-1}$$

$$N(f) = N_o$$

$$S/2N_o = 10^2$$

have shown that for $T \gtrsim 1$ second the error is again not greater than unity. Thus, this definition of dimensionality, although defined as a limit for $T \to \infty$, gives meaningful information for practical values of T.

CHAPTER IV
STUDY OF SUBOPTIMUM MODULATION TECHNIQUES

It has been shown that "optimum" digital communication over the channel of Fig. 1 involves use of the channel once for T seconds to transmit one of M signals. It has been demonstrated that "good" signals should be constructed as finite linear combinations of the first N eigenfunctions, i.e., each $x_j(t)$ is of the form

$$x_j(t) = \sum_{i=1}^{N} x_{ij}\varphi_i(t) \qquad 0 \leqslant t \leqslant T \tag{60}$$

and that the optimum detector makes a decision based on the N numbers y_i, where†

$$y_i = (y, \Theta_i)_\infty \quad . \tag{61}$$

It should be recalled, however, that there has been no claim that this approach is in any sense practical. In fact, since T might be on the order of hours, days, or weeks and since the generation of a large number of different functions [the $\{\varphi_i(t)\}$] of this duration is not feasible, it is clear that it is not. The purpose of this section is to investigate some "suitable substitutes" for the $\{\varphi_i(t)\}$ and to compare the resulting error exponents to that obtained when the $\{\varphi_i(t)\}$ are used. In this manner it will be possible to make an engineering evaluation of the trade-off between equipment complexity and performance.

In the "optimum" approach to digital communication over the channel of Fig. 1 there are two basic operations:

> The selection and generation of the transmitted signal $x_j(t)$;
> The receiver decision based on the channel output $y(t)$.

However, when considering an implementation of this approach it is convenient to break the problem into two different classifications, commonly called coding and modulation:[81]

> Under modulation is included the problem of generating the $\{\varphi_i(t)\}$, or suitable substitutes, and the problem of determining the y_i of Eq. (61).
>
> Under coding is included the problem of selecting the coefficients x_{ij} and the problem of making a decision based on the numbers y_i.

Although this breakdown is convenient and widely used in practice, it must be emphasized that the basic operations are the two indicated previously. Thus, in the design of an efficient communication system, coding and modulation must be considered together and possible trade-offs between the two evaluated. In practice, this might be accomplished by evaluating the cost and performance of various combinations of several coding and modulation techniques.

In view of the existence of several practical coding techniques[82-84] and the lack of significant previous studies of modulation for the channel of Fig. 1, the remainder of this report is

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† For simplicity of presentation, the following discussion assumes white noise and an infinite observation interval. However, unless otherwise indicated, the discussion is valid, with obvious modifications, for any noise spectrum and any observation interval by simply introducing the appropriate $\{\phi_i(t)\}$ and inner product for y_i from Theorems 1 to 3.

concerned with a theoretical investigation of several modulation techniques and an experimental investigation of one that appears quite promising for the telephone channel.

A basic problem in designing a modulation system is the determination of "suitable substitutes" for the $\{\varphi_i(t)\}$ of Eq. (60); it simply is not practical to think of constructing a large number of signals that may last for hours, days, or weeks. Thus, in practice it is necessary to replace the $\{\varphi_i(t)\}$ by a set of functions that are time translates of a small set of relatively simple signals. When this is done the phenomena called intersymbol interference appears due to the time-dispersive nature of the channel. As a result, the modulation problem reduces to a study of various techniques for overcoming intersymbol interference. The following sections consider several such techniques.

A. SIGNAL DESIGN TO ELIMINATE INTERSYMBOL INTERFERENCE

This section considers the possibility of substituting for the $\{\varphi_i(t)\}$ a set of time translates of a single, time-limited signal that has been designed to eliminate intersymbol interference. Although this problem can be formulated and a formal solution presented for an arbitrary observation interval, it is practical to obtain numerical results only for the infinite interval. In view of this and the fact that the resulting analysis is greatly simplified, an infinite observation interval ($T_1 = \infty$) is assumed at the outset. For simplicity, it is also assumed that the noise is white; the generalization to colored noise is presented in Appendix F.

Consider the situation in which a time-limited signal x(t) of $\mathcal{T}$ seconds duration is transmitted through the channel of Fig. 1.† Assume that the channel is followed by a matched filter whose output is sampled at $t = k\mathcal{T}$, $k = 0, \pm 1, \pm 2, \ldots$. Then by designing x(t) so that the matched filter output is zero for all sampling instants except $t = 0$, it will be possible to transmit a sequence of time translates (by $k\mathcal{T}$ seconds) of x(t) without incurring intersymbol interference. It is not clear, however, that this approach will lead to performance that is acceptable relative to the optimum performance found previously. This question can be investigated by considering first the problem of choosing a fixed energy x(t) to eliminate intersymbol interference and in addition give maximum energy at the channel output. It will then be possible to compare the error exponent for the best possible performance of this suboptimum technique with the optimum exponent found previously.

The problem of choosing x(t) to satisfy the conditions indicated above can be solved using standard techniques of the calculus of variations.[64,85] From Fig. 1 and the definition of a matched filter it follows that when x(t) is transmitted, the matched filter output for $t = k\mathcal{T}$ is

$$\int_{-\infty}^{\infty}\left[\int_0^{\mathcal{T}} x(\sigma)\, h(t-\sigma)\, d\sigma\right]\left[\int_0^{\mathcal{T}} x(\rho)\, h(t-k\mathcal{T}-\rho)\, d\rho\right] dt$$

$$= \int_0^{\mathcal{T}}\int_0^{\mathcal{T}} x(\sigma)\, x(\rho)\, R_h(\sigma-k\mathcal{T}-\rho)\, d\sigma\, d\rho$$

where

$$R_h(\tau) \triangleq \int_{-\infty}^{\infty} h(t)\, h(t-\tau)\, dt$$

† For this problem, $\mathcal{T}$ will be on the order of the reciprocal of the channel bandwidth.

and that the energy of x(t) at the channel output is simply the matched filter output for k = 0. Finally, the energy of x(t) is given by

$$\int_0^{\mathcal{T}} x^2(t)\,dt \quad .$$

Thus, by means of Lagrange multipliers, the constrained maximization problem requires maximization of the functional.

$$I = \int_0^{\mathcal{T}} \left\{ -\lambda x^2(\sigma) + \int_0^{\mathcal{T}} x(\sigma)\, x(\rho)\, R_h(\sigma-\rho)\, d\rho + \sum_{k=1}^{N} 2\beta_k \right.$$
$$\left. \times \int_0^{\mathcal{T}} x(\sigma)\, x(\rho)\, R_h(\sigma-\rho-k\mathcal{T})\, d\rho \right\} d\sigma \tag{62}$$

where the intersymbol interference constraints have been applied only to sampling times greater than zero since the output of a matched filter is an even function of time about t = 0. The number of successive sampling times N to which the constraints are applied is arbitrary at this point.

The maximization of I is accomplished in the usual manner by substituting $x(t) = \gamma(t) + \epsilon f(t)$, where $\gamma(t)$ is the desired solution, and setting

$$\left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} = 0 \quad .$$

Appendix C shows that this implies

$$\lambda\gamma(t) = \int_0^{\mathcal{T}} \gamma(\tau)\, K(t-\tau)\, d\tau \qquad 0 \leqslant t \leqslant \mathcal{T} \tag{63}$$

where

$$K(t-s) \triangleq R_h(t-s) + \sum_{k=1}^{N} \beta_k [R_h(t-s+k\mathcal{T}) + R_h(t-s-k\mathcal{T})] \quad .$$

Since $K(t-s)$ is symmetric and $\mathcal{L}_2$,† it is known[64] that a solution to this equation exists. However, to the author's knowledge, there are no results in the theory of integral equations which insure that the $\{\beta_k\}$ can be selected to satisfy the intersymbol interference constraints. Under the restriction that h(t) is the impulse response of a lumped parameter system, this difficulty can be overcome by transforming Eq. (63) into a differential equation with boundary conditions. (See Tricomi[65] for a discussion of the relation between boundary value differential equations and integral equations.) The boundary conditions will be determined first by investigating the properties that <u>any</u> signal must have to give zero intersymbol interference at the matched filter output.

† When all the β_k are finite, this follows directly from Appendix A.

By definition, the impulse response of a lumped parameter system can be written as

$$h(t) = \begin{cases} \sum_{i=1}^{n} a_i e^{-s_i t} & t \geqslant 0 \\ 0 & t < 0 \end{cases} \tag{64}$$

where a_i and s_i are, in general, complex constants satisfying certain conditions which insure that h(t) is real. Thus, for $t > \mathcal{T}$, the response r(t) to an input x(t) that is nonzero only on the interval $[0, \mathcal{T}]$ is

$$r(t) = \int_0^{\mathcal{T}} x(\sigma)\, h(t-\sigma)\, d\sigma = \sum_{i=1}^{n} a_i X(-s_i)\, e^{-s_i t}$$

where

$$X(-s_i) \triangleq \int_0^{\mathcal{T}} x(\sigma)\, e^{s_i \sigma}\, d\sigma \quad .$$

Similarly the matched filter output, say z(t), for $t \geqslant \mathcal{T}$ is given by

$$\begin{aligned} z(t) &= \int_0^{\infty} r(\sigma)\, r(t+\sigma)\, d\sigma \\ &= \sum_{i=1}^{n} a_i X(-s_i)\, e^{-s_i t} \int_0^{\infty} r(\sigma)\, e^{-s_i \sigma}\, d\sigma \\ &= \sum_{i=1}^{n} a_i X_i H(s_i)\, e^{-s_i t} \end{aligned}$$

where

$$H(s) \triangleq \int_0^{\infty} h(t)\, e^{-st}\, dt$$

and

$$X_i \triangleq X(s_i)\, X(-s_i) \quad .$$

Now, assume that intersymbol interference is zero for n successive sampling instants, i.e., $z(k\mathcal{T}) = 0$ for $k = 1, 2, \ldots, n$.† Then the $\{X_i\}$ must satisfy the n homogeneous equations

$$\sum_{i=1}^{n} a_i X_i H(s_i)\, e^{-s_i k \mathcal{T}} = 0 \qquad k = 1, 2, \ldots, n \quad .$$

† In order that the error exponent may be evaluated for these signals it is necessary that intersymbol interference be zero for all sampling instants, i.e., that $N = \infty$ in Eq. (62). However, the results of the following derivation are independent of N, provided $N \geqslant n$, and as demonstrated later, lead to signals giving zero intersymbol interference at all sampling instants. Thus, for convenience, $N = n$ is assumed at this point.

However, these equations can be satisfied by nonzero X_i if, and only if, the determinant of the coefficients of the $\{X_i\}$ vanishes. By forming this determinant and factoring out common terms, it reduces to

$$\begin{vmatrix} 1 & \cdots & 1 & \cdots & 1 \\ e^{-s_1\mathcal{T}} & & e^{-s_k\mathcal{T}} & & e^{-s_n\mathcal{T}} \\ (e^{-s_1\mathcal{T}})^2 & & \cdot & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & \cdot & & \cdot \\ (e^{-s_1\mathcal{T}})^{n-1} & \cdots & (e^{-s_k\mathcal{T}})^{n-1} & \cdots & (e^{-s_n\mathcal{T}})^{n-1} \end{vmatrix} .$$

This is a Vandermonde determinant and is known[86] to be equal to

$$\prod_{\substack{i=1 \\ k=1\ldots i-1}}^{n} \left(e^{-s_i\mathcal{T}} - e^{-s_k\mathcal{T}}\right) .$$

Thus, the determinant will be zero if, and only if, for some $i \neq k$

$$s_i = s_k + j\,\frac{2\pi\ell}{\mathcal{T}} \tag{65}$$

where

$$j = \sqrt{-1}$$

$$\ell = 0, \pm 1, \pm 2, \ldots \quad .$$

However, this relation between the poles of H(s) will not exist in general.† Thus, the $\{X_i\}$ must be identically zero if there is to be zero intersymbol interference at the matched filter output. Since $X_i = X(s_i)\,X(-s_i)$, it follows that for each $i = 1, 2, \ldots, n$ either $X(s_i) = 0$ or $X(-s_i) = 0$. [The condition $X(s_i) = 0$ <u>and</u> $X(-s_i) = 0$ is not included in the following discussion since it is a sufficient but not necessary condition for obtaining zero intersymbol interference.] Although not obvious, this restriction on x(t) is precisely what is required to obtain a solution to Eq. (63).

Recall that x(t) is defined to be nonzero only on the interval $[0, \mathcal{T}]$. Therefore, X(s) is an entire function which, from the above arguments, has zeros at either s_i or $-s_i$, $i = 1, \ldots, n$. Gerst and Diamond[87] have shown that a Laplace transform with these properties can be written as

$$X(s) = Q(s)\,V(s) \tag{66}$$

† The remainder of this discussion ignores the special cases in which Eq. (65) is satisfied since these are of limited practical interest.

where

$$Q(s) \triangleq \prod_{i=1}^{n} (s - z_i) \tag{67}$$

and $z_i = s_i$ or $-s_i$ according to whether $X(s_i) = 0$ or $X(-s_i) = 0$, respectively. V(s) is the Laplace transform of a pulse v(t) that has a continuous $(n-1)^{st}$ derivative and satisfies the boundary conditions

$$\begin{aligned} v(0) &= v(\mathcal{T}) = 0 \\ &\vdots \\ v^{(n-1)}(0) &= v^{(n-1)}(\mathcal{T}) = 0 \end{aligned} \tag{68}$$

where

$$v^{(i)}(t) \triangleq \frac{d^i v}{dt^i} \quad .$$

Thus, the result is obtained that any solution to Eq. (63) which satisfies the intersymbol interference constraints must be of the form

$$\gamma(t) = Q_t(p)\ v(t)$$

where

$$Q_t(p) \triangleq Q(s) \Big|_{s=p=d/dt}$$

and v(t) satisfies the integral equation

$$\lambda Q_t(p)\ v(t) = \int_0^{\mathcal{T}} [Q_\sigma(p)\ v(\sigma)]\ K(t-\sigma)\ d\sigma \qquad 0 \leqslant t \leqslant \mathcal{T} \tag{69}$$

as well as the boundary conditions of Eq. (68). This result provides the boundary conditions required to obtain a solution to Eq. (63). The corresponding differential equation will be derived next.

Observe that the operator $Q_\sigma(p)$ under the integral sign in Eq. (69) involves differentiation with respect to σ. Thus, as shown in Appendix D, integration by parts yields, after substituting the boundary conditions of Eq. (68),†

$$\lambda Q_t(p)\ v(t) = \int_0^{\mathcal{T}} v(\sigma)\ [Q_\sigma(-p)\ K(t-\sigma)]\ d\sigma \tag{70}$$

or

$$\lambda Q_t(p)\ v(t) = Q_t(p) \int_0^{\mathcal{T}} v(\sigma)\ K(t-\sigma)\ d\sigma \tag{71}$$

† The fact that $K(t-\sigma)$ is a linear combination of translates of a lumped parameter autocorrelation function implies that K(t) has derivatives of all orders when impulses and their derivatives are allowed.

where the last step follows from the fact that

$$\frac{d^i}{dt^i} K(t-\sigma) = (-1)^i \frac{d^i}{d\sigma^i} K(t-\sigma) \quad .$$

Next, observe from Eqs. (64) and (67) that the definition of Q(s) implies that

$$H(s)\, H(-s) \triangleq \frac{N(s^2)}{D(s^2)} = c\, \frac{N(s^2)}{Q(s)\, Q(-s)}$$

where $N(s^2)$ and $D(s^2)$ are polynomials in s^2 and c is a constant. Thus, applying the differential operator $Q_t(-p)$ to both sides of Eq. (71) gives

$$\lambda D(p^2)\, v(t) = D(p^2) \int_0^{\mathcal{T}} v(\sigma)\, K(t-\sigma)\, d\sigma \tag{72}$$

where

$$D(p^2) \triangleq D(s^2)\Big|_{s=p=d/dt} \quad .$$

Since, by definition,

$$K(t-\sigma) = \int_{-\infty}^{\infty} \frac{N(s^2)}{D(s^2)} \exp[s(t-\sigma)] \left[1 + \sum_{k=1}^{n} \beta_k (e^{sk\mathcal{T}} + e^{-sk\mathcal{T}})\right] df \qquad s = j2\pi f$$

Eq. (72) implies that

$$\begin{aligned}
\lambda D(p^2)\, v(t) &= D(p^2) \int_0^{\mathcal{T}} v(\sigma) \int_{-\infty}^{\infty} \frac{N(s^2)}{D(s^2)} \exp[s(t-\sigma)] \left[1 + \sum_{k=1}^{n} \beta_k (e^{sk\mathcal{T}} + e^{-sk\mathcal{T}})\right] df\, d\sigma \\
&= \int_0^{\mathcal{T}} v(\sigma) \int_{\infty}^{\infty} N(s^2) \exp[s(t-\sigma)] \left[1 + \sum_{k=1}^{n} \beta_k (e^{sk\mathcal{T}} + e^{-sk\mathcal{T}})\right] df\, d\sigma \\
&= N(p^2) \int_0^{\mathcal{T}} v(\sigma) \int_{-\infty}^{\infty} \exp[s(t-\sigma)] \left[1 + \sum_{k=1}^{n} \beta_k (e^{sk\mathcal{T}} + e^{-sk\mathcal{T}})\right] df\, d\sigma \\
&= N(p^2) \int_0^{\mathcal{T}} v(\sigma) \left\{\delta(t-\sigma) + \sum_{k=1}^{n} \beta_k [\delta(t-\sigma+k\mathcal{T}) + \delta(t-\sigma-k\mathcal{T})]\right\} d\sigma \\
&= N(p^2)\, v(t) \qquad 0 < t < \mathcal{T} \quad .
\end{aligned} \tag{73}$$

This differential equation together with the boundary conditions of Eq. (68) and the relation

$$\gamma(t) = Q_t(p)\, v(t) \tag{74}$$

defines the solution to Eq. (63) which satisfies the intersymbol interference constraints. However, since this is a boundary value instead of an initial value differential equation, there is no assurance from the previous work that a solution exists.[88] Brauer[89] recently studied this problem and

found that there exists a countably infinite discrete set of eigenvalues $\{\lambda_i\}$ for which there are corresponding eigenfunctions $\{v_i(t)\}$ that satisfy both the differential equation and the boundary conditions. Thus, a solution to the integral equation of Eq. (63) which satisfies the intersymbol interference constraints exists and satisfies Eqs. (68), (73), and (74).

Before proceeding with an investigation of the properties of the $\{v_i(t)\}$ and the corresponding $\{\gamma_i(t)\}$, two points should be noted. First, observe that the only property of the polynomial Q(s) used in deriving Eq. (73) from Eq. (71) was

$$Q(s)\, Q(-s) = cD(s^2) \quad . \tag{75}$$

Therefore, it is possible to arbitrarily choose either $z_i = s_i$ or $z_i = -s_i$, for $i = 1, \ldots, n$ in Eq. (67) without changing the fact that resulting $\{\gamma_i(t)\}$ are solutions of Eq. (63). Thus, there are 2^n possible choices for Q(s) each of which leads to an equally valid solution to the original maximization problem. Second, observe that Brauer's result states that there are an infinite number of solutions to Eqs. (68) and (73). Thus, the question arises as to which of these solutions is the one that yields maximum output signal energy. It is shown below that if the eigenvalues are ordered so that $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \ldots$, then $\gamma_1(t) = Q_t(p)\, v_1(t)$ is the desired optimum signal.

The fact that the $\{v_i(t)\}$ satisfy Eqs. (68) and (73) leads to several interesting and useful properties of the corresponding channel input signals $\{\gamma_i(t)\}$. One property of the $\{\gamma_i(t)\}$ is that they are orthogonal and may be assumed normalized. To verify this, let the following notation be adopted.

$$(f, g)_{\mathcal{T}} \triangleq \int_0^{\mathcal{T}} f(t)\, g(t)\, dt \tag{76}$$

and

$$Pf(t) \triangleq P(p)\, f(t)$$

where $P(p) = P(s)\big|_{s=p=d/dt}$ and P(s) is a polynomial. Then it follows that

$$(\gamma_i, \gamma_j)_{\mathcal{T}} = (Qv_i, Qv_j)_{\mathcal{T}} \quad .$$

Upon integrating by parts and substituting the boundary conditions of Eq. (68) this becomes†

$$(\gamma_i, \gamma_j)_{\mathcal{T}} = (v_i, Q^- Qv_j) = c(v_i, Dv_j) \tag{77}$$

where

$$Q^-(s) \triangleq Q(-s)$$

and the last step follows from Eq. (75). Now, from Eq. (73) it is found that

$$\lambda_i(v_j, Dv_i)_{\mathcal{T}} = (v_j, Nv_i)_{\mathcal{T}} \tag{78}$$

and also that

$$\lambda_j(v_i, Dv_j)_{\mathcal{T}} = (v_i, Nv_j)_{\mathcal{T}} \quad . \tag{79}$$

However, integration by parts and substitution of the boundary conditions of Eq. (68) shows that‡

† See Appendix D for the details of a similar integration by parts.

‡ Operators satisfying the following relations are said to be self-adjoint.

$$(v_j, Dv_i)_{\mathcal{T}} = (v_i, Dv_j)_{\mathcal{T}}$$

and

$$(v_j, Nv_i)_{\mathcal{T}} = (v_i, Nv_j)_{\mathcal{T}} \quad .$$

Thus, it follows from Eqs. (78) and (79) that

$$(\lambda_j - \lambda_i)\ (v_i, Dv_j)_{\mathcal{T}} = 0 \tag{80}$$

and therefore that $(v_i, Dv_j)_{\mathcal{T}} = 0$ if $(\lambda_j - \lambda_i) \neq 0$. From Eq. (77) it then follows that all $\gamma_i(t)$ corresponding to nondegenerate $v_i(t)$ are orthogonal.† Finally, since Eq. (73) is linear, it follows that the $\{\gamma_i(t)\}$ may be normalized; a convenient normalization, which is assumed throughout the remainder of this work, redefines the $\{\gamma_i(t)\}$ as

$$\sqrt{c}\ \gamma_i(t) = Q_t(p)\ v_i(t) \tag{81}$$

and assumes that

$$(\gamma_i, \gamma_i)_{\mathcal{T}} = 1 \quad . \tag{82}$$

A second property of the $\{\gamma_i(t)\}$ is that they are "doubly orthogonal" after passing through the channel filter in the sense that if $r_{ij}(t)$ is the filter output when $\gamma_i(t - j\mathcal{T})$ is transmitted then

$$(r_{ij}, r_{k\ell})_{\infty} = \begin{cases} \lambda_i & \text{if } i = k \text{ and } j = \ell \\ 0 & \text{otherwise} \end{cases} \quad . \tag{83}$$

In other words, the $\{\gamma_i(t)\}$ are orthogonal at the filter output and in addition, nonequal time translates of any two of the functions are orthogonal at the filter output.‡ This result can be verified by noting from Eqs. (76) and (81) that

$$(r_{ij}, r_{k\ell})_{\infty} = \frac{1}{c} \int_0^{\infty} \int_0^{\mathcal{T}} [Q_\sigma(p)\ v_i(\sigma)]\ h(t - j\mathcal{T} - \sigma)\ d\sigma \int_0^{\mathcal{T}} [Q_\rho(p)\ v_k(\rho)]\ h(t - \ell\mathcal{T} - \rho)\ d\rho\ dt$$

$$= \frac{1}{c} \int_0^{\mathcal{T}} \int_0^{\mathcal{T}} [Q_\sigma(p)\ v_i(\sigma)]\ [Q_\rho(p)\ v_k(\rho)]\ R_h[\sigma - \rho + (j - \ell)\mathcal{T}]\ d\sigma\, d\rho \quad .$$

Integration by parts and substitution of the boundary conditions of Eq. (68) show that this becomes

$$(r_{ij}, r_{k\ell})_{\infty} = \frac{1}{c} \int_0^{\mathcal{T}} \int_0^{\mathcal{T}} v_i(\sigma)\ v_k(\rho)\ \{Q_\sigma(-\rho)\ Q_\sigma(p) R_h\ [\sigma - \rho + (j - \ell)\mathcal{T}]\}\ d\sigma\, d\rho$$

or from Eq. (75),

$$(r_{ij}, r_{k\ell})_{\infty} = \frac{1}{c} \int_0^{\mathcal{T}} \int_0^{\mathcal{T}} v_i(\sigma)\ v_k(\rho)\ \{D(p^2) R_h\ [\sigma - \rho + (j - \ell)\mathcal{T}]\}\ d\sigma\, d\rho \quad .$$

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† It can be shown that degenerate eigenfunctions are of a finite multiplicity and may be assumed to be orthogonal.[89]

‡ It should be observed that the orthogonality of time translates of the $\{\gamma_i(t)\}$ follows directly from the fact that the $\{\gamma_i(t)\}$ have been chosen to give zero intersymbol interference. In other words, the conditions of orthogonality at the channel filter output and zero intersymbol interference at the matched filter output are identical. Thus, the $\{\gamma_i(t)\}$ give zero intersymbol interference at all sampling instants.

However, since

$$R_h(\sigma-\rho) = \int_{-\infty}^{\infty} \frac{N(s^2)}{D(s^2)} \exp[s(\sigma-\rho)]\, df \qquad s = j2\pi f$$

this implies that

$$(r_{ij}, r_{k\ell})_\infty = \int_0^{\mathcal{T}}\int_0^{\mathcal{T}} v_i(\sigma)\, v_k(\rho) \left(D(p^2) \int_{-\infty}^{\infty} \frac{N(s^2)}{D(s^2)} \exp\{s[\sigma-\rho+(j-\ell)\mathcal{T}]\}\, df\right) d\sigma\, d\rho$$

$$= \int_0^{\mathcal{T}}\int_0^{\mathcal{T}} v_i(\sigma)\, v_k(\rho) \left(N(p^2) \int_{-\infty}^{\infty} \exp\{s[\sigma-\rho+(j-\ell)\mathcal{T}]\}\, df\right) d\sigma\, d\rho$$

$$= \int_0^{\mathcal{T}}\int_0^{\mathcal{T}} v_i(\sigma)\, v_k(\rho)\, \{N(p^2)\, \delta\, [\sigma-\rho+(j-\ell)\mathcal{T}]\}\, d\sigma\, d\rho$$

$$= \begin{cases} (v_i, Nv_k)_{\mathcal{T}} & \text{if } j = \ell \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \lambda_j (v_i, Dv_k)_{\mathcal{T}} & \text{if } j = \ell \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \lambda_i & \text{if } i = k \text{ and } j = \ell \\ 0 & \text{otherwise} \end{cases} \tag{84}$$

where the last two lines follow from Eqs. (79) to (82). Thus, the $\{\gamma_i(t)\}$ are "doubly orthogonal" in the sense stated.

A third and extremely important property of the $\{\gamma_i(t)\}$ is that the corresponding eigenvalues $\{\lambda_i\}$ are the ratio of the output to input signal energy, i.e.,

$$\lambda_i = \frac{(r_{i0}, r_{i0})_\infty}{(\gamma_i, \gamma_i)_{\mathcal{T}}} .$$

This is readily verified by noting from Eqs. (77), (81), and (84) that

$$\frac{(r_{i0}, r_{i0})_\infty}{(\gamma_i, \gamma_i)_{\mathcal{T}}} = \frac{(v_i, Nv_i)_{\mathcal{T}}}{(v_i, Dv_i)_{\mathcal{T}}} = \lambda_i . \tag{85}$$

Thus, with $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \dots$, it follows that $\gamma_1(t)$ is the solution to the original maximization problem.

A final property of the $\{\gamma_i(t)\}$ is obtained for the special case in which $z_i = -s_i$ for all i in Eq. (67). For this case it follows directly that

$$\gamma_i(t) = D^+(p)\, v_i(t)$$

where

$$H(s) \triangleq \frac{N^+(s)}{D^+(s)}$$

and thus that

$$r_{i0}(t) = N^{+}(p)\, v_i(t) \quad .$$

Since $N^{+}(p)\, v_i(t)$ is a linear combination of derivatives of $v_i(t)$, this demonstrates that $r_{i0}(t)$ is time limited to $[0, \mathcal{T}]$, i.e., the time-limited input $\gamma_i(t)$ yields an output $r_{i0}(t)$ that is again time limited to the same interval.† This special form of the solution to Eq. (63) has been studied by Hancock and Schwarzlander[90] and is of practical interest due to the possible simplification of matched filter construction for time-limited signals.

As an illustration of these results, consider the situation in which the channel filter is given by

$$H(s) = \frac{1}{1+s} \quad .$$

Then

$$\frac{N(s^2)}{D(s^2)} \triangleq H(s)\, H(-s) = \frac{1}{1-s^2}$$

and the boundary value differential equation becomes

$$\left[1 - \lambda_i\left(1 - \frac{d^2}{dt^2}\right)\right] v_i(t) = 0$$

or

$$\left(\frac{d^2}{dt^2} + \omega_i^2\right) v_i(t) = 0 \qquad 0 < t < \mathcal{T}$$

where

$$\omega_i^2 = \frac{1}{\lambda_i} - 1$$

and

$$v_i(0) = v_i(\mathcal{T}) = 0 \quad .$$

From this it follows readily that

$$v_i(t) = \begin{cases} a_i \sin \omega_i t & 0 \leqslant t \leqslant \mathcal{T} \\ 0 & \text{elsewhere} \end{cases}$$

where

$$\omega_i = i\pi/\mathcal{T} \qquad\qquad i = 1, 2, 3, \ldots \quad .$$

From Eq. (67) and the discussion following Eq. (75), Q(s) is

$$Q(s) = 1 \pm s \quad .$$

† To the author's knowledge, this property of the solutions of Eq. (63) was first recognized by Richters.[91]

Thus, the eigenfunctions and eigenvalues are

$$\lambda_i = [1 + \omega_i^2]^{-1} \qquad i = 1, 2, \ldots$$

and

$$\gamma_i(t) = \left[\frac{2\lambda_i}{\mathcal{T}}\right]^{1/2} [\sin \omega_i t \pm \omega_i \cos \omega_i t] \quad .$$

From this observe that

$$\int_0^{\mathcal{T}} \gamma_i(t)\, \gamma_j(t)\, dt = \left[\frac{4\lambda_i\lambda_j}{\mathcal{T}^2}\right]^{1/2} \int_0^{\mathcal{T}} [\sin \omega_i t \sin \omega_j t + \omega_i\omega_j \cos \omega_i t \cos \omega_j t$$

$$\pm\, \omega_i \cos \omega_i t \sin \omega_j t \pm \omega_j \sin \omega_i t \cos \omega_j t]\, dt \quad .$$

However,

$$\int_0^{\mathcal{T}} \sin \omega_i t \sin \omega_j t\, dt = \int_0^{\mathcal{T}} \cos \omega_i t \cos \omega_j t\, dt = \frac{\mathcal{T}}{2}\, \delta_{ij}$$

and integration by parts shows that

$$\omega_i \int_0^{\mathcal{T}} \cos \omega_i t \sin \omega_j t\, dt = -\omega_j \int_0^{\mathcal{T}} \sin \omega_i t \cos \omega_j t\, dt \quad .$$

Thus,

$$\int_0^{\mathcal{T}} \gamma_i(t)\, \gamma_j(t)\, dt = \left[\frac{4\lambda_k\lambda_j}{\mathcal{T}^2}\right]^{1/2} \frac{\mathcal{T}}{2}\, [1 + \omega_i^2]\, \delta_{ij} = \delta_{ij}$$

and the $\{\gamma_i(t)\}$ are orthonormal. Next, consider

$$r_{ij}(t) \triangleq \int_{-\infty}^{\infty} \gamma_i(\sigma - j\mathcal{T})\, h(t - \sigma)\, d\sigma \quad .$$

Evaluation of the integral shows that

$$r_{ij}(t) = \lambda_i \left[\frac{2\lambda_i}{\mathcal{T}}\right]^{1/2} \begin{cases} 0 & t < j\mathcal{T} \\ (1 \pm \omega_i^2) \sin \omega_i (t - j\mathcal{T}) - \omega_i (1 \mp 1) \{\cos \omega_i (t - j\mathcal{T}) - \exp[-(t - j\mathcal{T})]\} & \\ & j\mathcal{T} \leqslant t \leqslant (j+1)\mathcal{T} \\ \omega_i (1 \mp 1) \exp[-(t - j\mathcal{T})]\, [(-1)^{i-1} e^{\mathcal{T}} + 1] & t > (j+1)\mathcal{T} \quad . \end{cases}$$

Observe from this that if the + sign is taken in the definition of Q(s), $r_{ij}(t)$ has the simple form

$$r_{ij}(t) = \left[\frac{2\lambda_i}{\mathcal{T}}\right]^{1/2} \begin{cases} \sin \omega_i (t - j\mathcal{T}) & j\mathcal{T} \leqslant t \leqslant (j+1)\mathcal{T} \\ 0 & \text{elsewhere} \quad . \end{cases}$$

In other words, the time-limited input signal $\gamma_i(t - j\mathcal{T})$ yields an output that is again time limited to the same interval. Furthermore, it follows almost trivially that

$$(r_{ij}, r_{k\ell})_\infty = \begin{cases} \lambda_i & \text{if } i = k \text{ and } j = \ell \\ 0 & \text{otherwise} \end{cases}$$

i.e., for this Q(s) the $\{\gamma_i(t)\}$ are "doubly orthogonal" at the filter output.† Also, note that since the $\{\gamma_i(t)\}$ are normalized, λ_i is the ratio of the output to input signal energy.

The previous work has derived the optimum time-limited signal that gives zero intersymbol interference at the matched filter output. It is now of interest to compare the performance of this modulation technique to that obtained when the optimum technique of Sec. III-F is used. Recall that optimum signals are constructed as linear combinations of the first N eigenfunctions, i.e., each signal is of the form

$$x_j(t) = \sum_{i=1}^{N} x_{ij}\varphi_i(t) \qquad 0 \leqslant t \leqslant T \qquad j = 1, \ldots, M$$

and that a study of suboptimum modulation techniques involves finding "suitable substitutes" for the $\{\varphi_i(t)\}$. For the suboptimum modulation technique of this section, the $\{\varphi_i(t)\}$ are replaced by time translates of the optimum signal $\gamma_1(t)$, i.e., if $\{\varphi_i'(t)\}$ are the functions substituted for the $\{\varphi_i(t)\}$ then‡

$$\varphi_i'(t) \triangleq \gamma_1(t - i\mathcal{T}) \qquad 0 \leqslant t \leqslant T \qquad i = 1, 2, \ldots, K \tag{86}$$

and a general input signal is of the form

$$x(t) = \sum_{i=1}^{K} x_i\varphi_i'(t) \qquad 0 \leqslant t \leqslant T \quad . \tag{87}$$

For this input, the corresponding channel output is

$$y(t) = \sum_{i=1}^{K} \sqrt{\lambda_1}\, x_i\Theta_i'(t) + n(t) \tag{88}$$

where

$$\Theta_i'(t) \triangleq \frac{1}{\sqrt{\lambda_1}} \int_0^T \varphi_i'(\tau)\, h(t - \tau)\, d\tau$$

and λ_1 is the first eigenvalue of Eq. (73). From this it follows that the matched filter output at $t = k\mathcal{T}$, $z(k\mathcal{T})$, is§

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† Verification of this property for $Q(s) = 1 - s$ is possible, but tedious and is omitted.

‡ It is assumed here that $\mathcal{T}$ and T are related by $T = K\mathcal{T}$ where K is an integer. However, K = N is not assumed since N is the optimum dimensionality of the $\{x_i(t)\}$ only when the $\{\phi_i(t)\}$ are used.

§ For convenience it is assumed here that the matched filter is matched to $\lambda_1^{-1/2}$ times the channel filter output when $\gamma_1(t)$ is transmitted. This, together with the assumed noise spectrum normalization (see Fig. 1) gives unity noise variance at the matched filter output.

$$z(k\mathcal{T}) = \int_0^\infty y(t)\ \Theta_1'(t - k\mathcal{T})\ dt = \int_0^\infty y(t)\ \Theta_k'(t)\ dt = (y, \Theta_k')_\infty$$

$$= \sum_{i=1}^{K} \sqrt{\lambda_1}\ x_i(\Theta_i', \Theta_k')_\infty + (n, \Theta_k')_\infty$$

$$= \sqrt{\lambda_1}\ x_k + n_k \tag{89}$$

where $n_k \triangleq (n, \Theta_k')_\infty$ and the last step follows from Eq. (84) by noting that $\sqrt{\lambda_1}\ \Theta_i'(t) = r_{1i}(t)$. Furthermore,

$$\overline{n_j n_k} = \int_0^\infty \int_0^\infty \overline{n(\sigma)\ n(\rho)}\ \Theta_k'(\sigma)\ \Theta_j'(\rho)\ d\sigma\, d\rho$$

$$= \int_0^\infty \int_0^\infty \delta(\sigma - \rho)\ \Theta_k'(\sigma)\ \Theta_j'(\rho)\ d\sigma\, d\rho$$

$$= (\Theta_k', \Theta_j')_\infty = \delta_{kj} \quad . \tag{90}$$

Thus, by comparing Eqs. (89) and (90) to the vector representations of Theorems 1 to 3, it follows that the value of C for this modulation technique,† say C', can be obtained from Eqs. (32) and (35) by substituting

$$\lambda_i = \begin{cases} \lambda_1' & i = 1, \ldots, K \\ 0 & \text{otherwise} \end{cases}$$

and performing a maximization over $\mathcal{T}$. [The prime has been used here to indicate that λ_1' is an eigenvalue of Eq. (73).] The resulting expression is

$$C' = \max_{\mathcal{T}} \frac{1}{2\mathcal{T}} \ln\,[1 + \lambda_1(\mathcal{T})\ S\mathcal{T}] \tag{91}$$

where the fact that λ_1 is a function of the length of the interval over which $\gamma_1(t)$ is defined has been emphasized by writing $\lambda_1 = \lambda_1(\mathcal{T})$. It should be mentioned that the maximization over $\mathcal{T}$ is required since the value of $\mathcal{T}$ in Eq. (86) is arbitrary and therefore should be chosen to maximize C'.

From Eq. (91) it is clear that a knowledge of the first eigenvalue of Eq. (73) is required as a function of the interval length $\mathcal{T}$. However, boundary value equations such as this are quite tedious to solve for specific cases. Thus, since only the eigenvalue is of interest and not the eigenfunction, it is desirable to investigate a means for determining the eigenvalue without solving the differential equation. Such a result can be obtained by considering the Laplace transform of the solution to Eqs. (68) and (73) repeated below for convenience

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† Recall that a conclusion of Sec. III-E was that a knowledge of R_c and C as a function of signal power S provides sufficient information about the error bounds for engineering purposes. However, in a study of suboptimum modulation systems which involves only a comparison (and not a numerical evaluation) of the error exponent for various techniques, it is convenient to make a further simplification and to evaluate only C' as a function of S; the tacit assumption being that if C' is close to C, then R_c' is close to R_c and conversely, if C' differs greatly from C then R_c' differs greatly from R_c.

$$[N(p^2) - \lambda_i D(p^2)]\, v_i(t) = 0 \qquad 0 < t < \mathcal{T} \tag{92}$$

$$v_i(0) = v_i(\mathcal{T}) = 0$$

$$\vdots$$

$$v_i^{(n-1)}(0) = v_i^{(n-1)}(\mathcal{T}) = 0 \quad . \tag{93}$$

Since no additional effort is required to obtain an expression which defines all the eigenvalues, this will be done.

Recall that by assumption the $\{v_i(t)\}$ are zero outside the interval $[0, \mathcal{T}]$ and that the polynomial $D(s^2)$ is of order 2n. Assume that the order of $N(s^2)$ is 2m and that $m \leqslant n$. Finally, observe that although the boundary conditions imply that all derivatives of $v_i(t)$ through the $(n-1)^{st}$ are zero at $t = 0$ and $t = \mathcal{T}$, it is possible that higher order derivatives of $v_i(t)$ will involve impulses and their derivatives at these end points. In view of these conditions, Eqs. (92) and (93) can be replaced by the equation

$$[N(p^2) - \lambda_i D(p^2)]\, v_i(t) = P_{1i}(p)\, \delta(t) + P_{2i}(p)\, \delta(t - \mathcal{T}) \qquad -\infty < t < \infty \tag{94}$$

where $P_{1i}(s)$ and $P_{2i}(s)$ are polynomials of order $(n-1)$ or less whose coefficients will be determined later. Since Eq. (94) is satisfied for all t, the Laplace transform of both sides can be taken. This yields

$$V_i(s) = \frac{P_{1i}(s) + P_{2i}(s)\, e^{s\mathcal{T}}}{N(s^2) - \lambda_i D(s^2)} \tag{95}$$

where $V_i(s)$ is the transform of $v_i(t)$ and the domain of convergence is the entire finite s-plane. It is shown in Appendix E that all nondegenerate solutions of Eqs. (92) and (93) are either even or odd functions about $t = \mathcal{T}/2$. Thus, since an even or odd function has an even or odd transform, it follows that for nondegenerate eigenfunctions

$$\frac{P_{1i}(s)\, e^{-s\mathcal{T}/2} + P_{2i}(s)\, e^{s\mathcal{T}/2}}{N(s^2) - \lambda_i D(s^2)} = \pm \frac{P_{1i}(-s)\, e^{s\mathcal{T}/2} + P_{2i}(-s)\, e^{-s\mathcal{T}/2}}{N(s^2) - \lambda_i D(s^2)}$$

or

$$P_{1i}(s) \pm P_{2i}(-s) = \pm[P_{1i}(-s) \pm P_{2i}(s)]\, e^{s\mathcal{T}} \quad . \tag{96}$$

Expansion of $e^{s\mathcal{T}}$ shows that the right-hand side of Eq. (96) is an infinite order polynomial while the left-hand side is at most an $(n-1)^{st}$ order polynomial. Thus,

$$P_{1i}(s) \pm P_{2i}(-s) = 0$$

and Eq. (95) becomes†

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† Although the previous discussion has shown Eq. (97) to be true only for nondegenerate eigenfunctions, it is possible by means of an argument identical to that used by Youla[92] to show that it is true for all eigenfunctions.

$$V_i(s) = \frac{P_{1i}(s) \pm P_{1i}(-s)\, e^{s\mathcal{T}}}{N(s^2) - \lambda_i D(s^2)} \quad . \tag{97}$$

Next, recall that $V_i(s)$ must be an entire function since $v_i(t)$ is a pulse. Thus, the coefficients in $P_{1i}(s)$ must be chosen so that the numerator of Eq. (97) contains all the zeros of the denominator.

If $\pm s_{i1}, \pm s_{i2}, \ldots, \pm s_{in}$ are the 2n roots of $N(s^2) - \lambda_i D(s^2) = 0$, this condition leads to the n homogeneous equations

$$P_{1i}(s_{i\ell}) \pm P_{1i}(-s_{i\ell})\, e^{s_{i\ell}\mathcal{T}} = 0 \qquad \ell = 1, \ldots, n \quad . \tag{98}$$

Since

$$P_{1i}(s) \triangleq \sum_{j=1}^{n} a_{ij}\, s^{j-1}$$

Eq. (98) becomes

$$a_{i1}(1 \pm w_{i\ell}) + a_{i2}(1 \mp w_{i\ell})\, s_{i\ell} + a_{i3}(1 \pm w_{i\ell})\, s_{i\ell}^2 + \ldots + a_{in}\,[1 \pm (-1)^{n-1} w_{i\ell}]\, s_{i\ell}^{n-1} = 0$$

$$\ell = 1, \ldots, n \tag{99}$$

where $w_{i\ell} \triangleq \exp[s_{i\ell}\mathcal{T}]$. Thus, a set of nonzero coefficients $\{a_{i\ell}\}$ exists if and only if

$$\Delta = \begin{vmatrix} (1 \pm w_{i1}) & (1 \mp w_{i1})\, s_{i1} & \ldots & [1 \pm (-1)^{n-1} w_{i1}]\,(s_{i1})^{n-1} \\ (1 \pm w_{i2}) & \ldots & \ldots & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ (1 \pm w_{in}) & (1 \mp w_{in})\, s_{in} & \ldots & [1 \pm (-1)^{n-1} w_{in}]\,(s_{in})^{n-1} \end{vmatrix} = 0 \tag{100}$$

Equation (100) is the desired result which allows evaluation of the $\{\lambda_i\}$ without explicit solution of the boundary value differential equation. [In addition, observe that this result and Eqs. (97) and (99) provide a frequency domain solution for the $\{v_i(t)\}$.]

The result will now be used to evaluate the performance of the suboptimum modulation technique of this section. A convenient class of channels to consider are those having Butterworth filter characteristics, i.e., channels for which

$$H(s)\, H(-s) = \frac{1}{1 + (-1)^n \left(\frac{s}{2\pi}\right)^{2n}} \quad .$$

For this class of filters, the roots of $N(s^2) - \lambda_i D(s^2) = 0$ can be expressed in the simple form

$$s_{i\ell} = (2\pi)\,[\lambda_i^{-1} - 1]^{1/2n} \exp\left[j\pi\left(\frac{\ell - 1}{n} + \frac{1}{2}\right)\right] \qquad \ell = 1, \ldots, n$$

Observe that if λ_i is written as

$$\lambda_i = \frac{1}{1 + \left(\frac{k_{ni}}{\mathcal{T}}\right)^{2n}} \qquad (101)$$

where k_{ni} is a constant, then

$$s_{i\ell} = 2\pi \frac{k_{ni}}{\mathcal{T}} \exp\left[j\pi\left(\frac{\ell - 1}{n} + \frac{1}{2}\right)\right] \qquad \ell = 1, \ldots, n$$

and after factoring out common terms, Eq. (100) becomes

$$\Delta = \begin{vmatrix} (1 \pm w_{i1}) & (1 \mp w_{i1}) & \cdots & [1 \pm (-1)^{n-1} w_{i1}] \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ (1 \pm w_{in}) & (1 \mp w_{in}) \{\exp[i\pi(n-1)/n]\} & \cdots & [1 \pm (-1)^{n-1} w_{in}] \{\exp[j\pi(n-1)/n]\}\ n-1 \end{vmatrix} = 0 \qquad (102)$$

where now $w_{i\ell} = \exp\{2\pi k_{ni} \exp[j\pi(\ell - 1/n) + (1/2)]\}$. Thus, for the Butterworth filters, Δ is independent of $\mathcal{T}$ when the $\{\lambda_i\}$ are expressed as in Eq. (101).

Consider first the case for $n = 1$. From Eq. (102) it follows that k_{1i} must satisfy the relation

$$\{1 \pm \exp[j2\pi k_{1i}]\} = 0 \quad . \qquad (103)$$

However, as noted previously, λ_i is the energy transfer ratio of the filter when $\gamma_i(t)$ is transmitted. Since the filter is normalized so that

$$\max_f |H(f)| = 1 \quad ,$$

it follows that $\lambda_i < 1$. Thus, $k_{1i} = 0$ is not an allowed solution to Eq. (103) and the final result is

$$k_{1i} = i/2 \qquad i = 1, 2, \ldots$$

or

$$\lambda_i = \frac{1}{1 + \left(\frac{i}{2\mathcal{T}}\right)^2} \qquad (104)$$

which agrees with the result found on page 67 after the introduction of a bandwidth scale factor of 2π. Substitution of this expression into Eq. (91) shows that

$$C' = \max_{\mathcal{T}} \frac{1}{2\mathcal{T}} \log_2 \left[1 + \frac{S\mathcal{T}}{1 + \left(\frac{i}{2\mathcal{T}}\right)^2}\right] \text{ bits/second} \quad .$$

This expression is plotted in Fig. 10 along with the value of $C = R(0)$. It should be noted that this technique is inferior to the optimum technique by approximately 3 db for $S/2 = 10$ and by nearly 7 db for $S/2 = 10^3$.

To investigate the performance for higher order channels, it is necessary to solve for k_{ni} from Eq. (102). Since this becomes quite tedious manually for $n > 2$, the evaluation has been accomplished numerically on a digital computer. The result is that to two significant figures and (at least) for $n \leqslant 10$, k_{ni} is given by

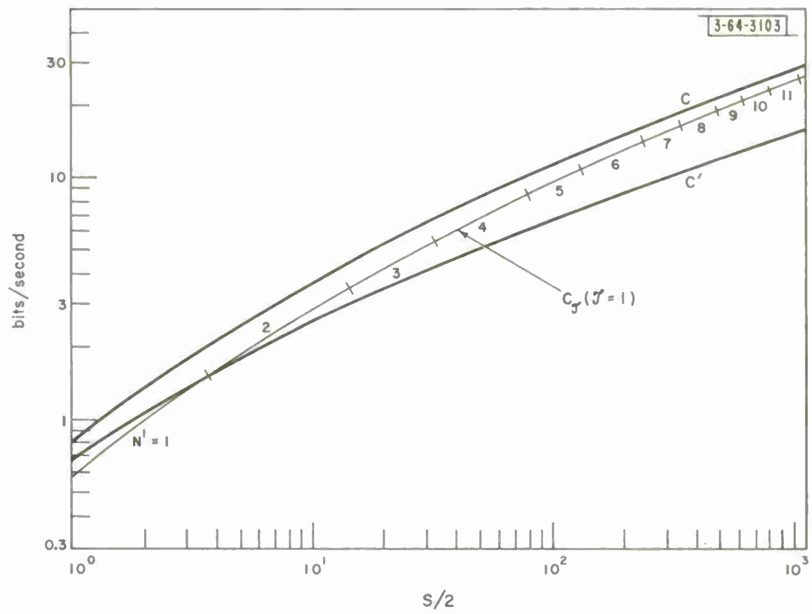


Fig. 10. C, C', and $C_{\mathcal{T}}$ for channel with N(f) = 1 and $|H(f)|^2 = [1 + f^2]^{-1}$.

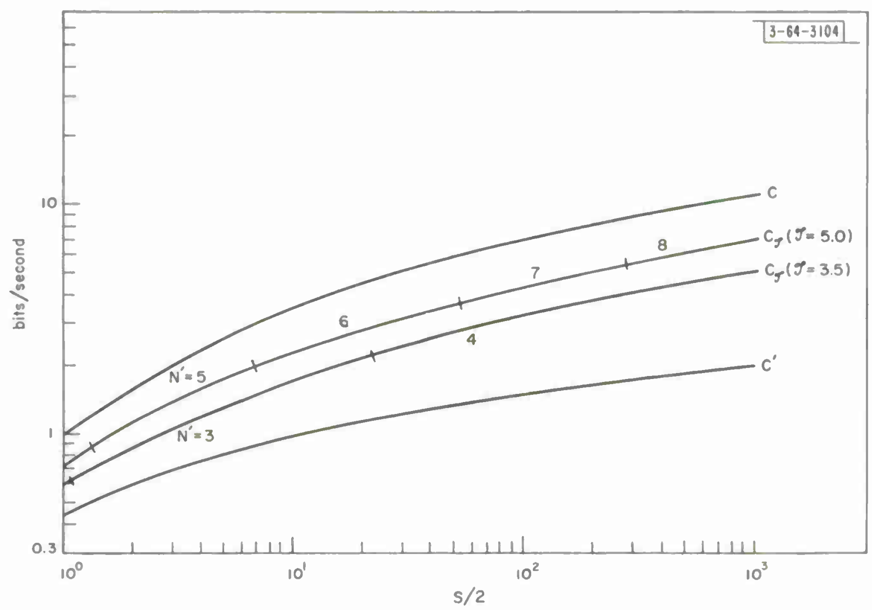


Fig. 11. C, C', and $C_{\mathcal{T}}$ for channel with N(f) = 1 and $|H(f)|^2 = [1 + f^{20}]^{-1}$.

$$k_{ni} = \frac{1}{2}\left[\frac{1}{2}(n-1) + i\right] \qquad i = 1, 2, \ldots \quad .$$

Thus,

$$\lambda_i = \left\{1 + \left[\frac{\frac{1}{2}(n-1) + i}{2\mathcal{T}}\right]^{2n}\right\}^{-1} \quad . \tag{105}$$

and Eq. (91) becomes

$$C' = \max_{\mathcal{T}} \frac{1}{2\mathcal{T}} \log_2 \{1 + S\mathcal{T} [1 + (\frac{n+1}{4\mathcal{T}})^{2n}]^{-1}\} \text{ bits/second} \quad .$$

Figure 11 presents C' for a Butterworth channel with $n = 10$ together with the value of $C = R(0)$ calculated from Eqs. (37) to (39). It should be noted that although $C' \to C$ for $S \to 0$, there is an effective signal power loss of 25 db for $S/2 = 10^3$.

This extremely poor performance at only moderately large signal powers when coupled with the lesser but still significant loss for the simple $n = 1$ channel suggests that this suboptimum modulation technique is of limited practical interest. The following paragraph considers an extension of the present technique that leads to significantly improved performance at the expense of an increase in equipment complexity.

Recall from Eqs. (80) and (84) that the $\{\gamma_i(t)\}$ are orthonormal and are "doubly orthogonal" at the channel filter output. Because of these properties it is possible to substitute for the $\{\varphi_i(t)\}$ not just time translates of the single function $\gamma_1(t)$ but instead time translates of the first few of the $\{\gamma_i(t)\}$. When this is done, $\mathcal{T}$ can be increased and improved performance obtained. More specifically, consider the following situation. Let $\mathcal{T}$ be fixed, assume that $T = k\mathcal{T}$ with k an integer, let the $\{\varphi_i(t)\}$ be replaced with time translates of the first N' of the $\{\gamma_i(t)\}$, and let N' be chosen to maximize the error exponent. It then follows from the discussion of Eq. (91) and Eqs. (32) and (35) that the value of C for this modulation technique, say $C_{\mathcal{T}}$, is

$$C_{\mathcal{T}} = \frac{1}{2\mathcal{T}} \sum_{i=1}^{N'} \log_2 \frac{\lambda_i}{B_{\mathcal{T}}(0)} \text{ bits/second} \tag{106}$$

where

$$\frac{1}{B_{\mathcal{T}}(0)} \triangleq \frac{S\mathcal{T} + \sum_{i=1}^{N'} \lambda_i^{-1}}{N'}$$

and

$$\lambda_{N'} > B_{\mathcal{T}}(0) \geqslant \lambda_{N'+1}$$

and the $\{\lambda_i\}$ are eigenvalues of Eqs. (92) and (93). Equation (106) is plotted in Figs. 10 and 11 for the $n = 1$ and $n = 10$ Butterworth channel, respectively, using the eigenvalues of Eq. (105). Observe from Fig. 10 that for the $n = 1$ channel and with $\mathcal{T} = 1$, use the first twelve of the $\{\gamma_i(t)\}$ gives a modulation system whose performance is within 1 db of ideal at $S/2 = 10^3$. Figure 11 demonstrates a similar improvement for the $n = 10$ channel except that for this channel both $\mathcal{T}$ and N' must be significantly greater than for the $n = 1$ channel to achieve the same level of performance.

In conclusion, two points should be made concerning this modulation technique.

(1) As suggested in Fig. 11, it is possible by using increasingly large values of $\mathcal{T}$ and N' to obtain a modulation system for which $C_{\mathcal{T}}$ is arbitrarily close to C. As a practical matter, it appears that a value of $\mathcal{T}$ between one and twenty times the reciprocal of the filter 3-db bandwidth is sufficient to achieve most of the improvement possible.

(2) Since the determination of the $\{\gamma_i(t)\}$ becomes increasingly difficult for higher order channels, and since a greater number of the $\{\gamma_i(t)\}$ are required for efficient operation with these channels, it is clear that this modulation technique can be considered practical only for low-order channels.

B. RECEIVER FILTER DESIGN TO ELIMINATE INTERSYMBOL INTERFERENCE

The previous section has considered a suboptimum modulation technique that eliminates intersymbol interference by means of a suitable choice of the transmitted signal. This section considers an alternate approach in which the receiver matched filter is replaced by a filter that has been designed to maximize SNR and eliminate intersymbol interference.†

Consider again the channel of Fig. 1, let N(f) be arbitrary, and assume that a signal x(t) is given. For this situation it is desired to design a receiver filter $h_1(t)$ so that when x(t) is transmitted the filter output will be zero at $t = k\mathcal{T}$, $k = \pm 1, \ldots, \pm N$, and nonzero at $t = 0$.‡ In this manner, it will be possible to transmit time translates (by $k\mathcal{T}$ seconds) of x(t) without incurring intersymbol interference. Since there are, in general, an infinite number of filters having this property, it is desirable to choose the filter that maximizes the SNR at $t = 0$. This problem is readily solved using standard techniques of the calculus of variations.[64,85] When x(t) is transmitted, the signal portion of the output of $h_1(t)$, say z(t) is

$$z(t) = \int_{-\infty}^{\infty} X(f)\, H(f)\, H_1(f)\, e^{j\omega t}\, df \tag{107}$$

where X(f), H(f), and $H_1(f)$ are the Fourier transforms of x(t), h(t), and $h_1(t)$, respectively. Also, the noise output, say $n_o(t)$, is

$$n_o(t) = \int_{-\infty}^{\infty} n(\sigma)\, h_1(t-\sigma)\, d\sigma \quad .$$

For this problem, a useful SNR definition is

$$\mathrm{SNR} = \frac{z^2(0)}{\overline{[n_o(t)]^2}} = \frac{\left[\int_{-\infty}^{\infty} X(f)\, H(f)\, H_1(f)\, df\right]^2}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{n(\sigma)\, n(\rho)}\, h(t-\sigma)\, h(t-\rho)\, d\sigma\, d\rho} = \frac{\left[\int_{-\infty}^{\infty} X(f)\, H(f)\, H_1(f)\, df\right]^2}{\int_{-\infty}^{\infty} N(f)\, |H_1(f)|^2\, df} \tag{108}$$

† To the author's knowledge, this problem was first considered by Tufts.[93] The work presented here represents an alternate and somewhat simplified derivation of his result and in addition provides some insight into the SNR degradation caused by the elimination of intersymbol interference.

‡ The number N is arbitrary at this point. Appropriate values will be indicated later when specific examples are considered.

Thus, the problem is that of choosing $H_1(f)$ to maximize Eq. (108) under the constraints $z(k\mathcal{T}) = 0$, $k = \pm 1, \ldots, \pm N$. Use of Lagrange multipliers shows that $H_1(f)$ must be chosen to minimize the functional†

$$I = \int_{-\infty}^{\infty} N(f) \left[|H_1(f)|^2 - X(f)\, H(f)\, H_1(f) \sum_{k=-N}^{N} 2\beta_k\, e^{j\omega k\mathcal{T}} \right] df \quad .$$

This minimization is accomplished in the usual manner by substituting $H_1(f) = H_o(f) + \epsilon H_2(f)$, where $H_o(f)$ is the desired optimum filter, and setting

$$\left.\frac{dI}{d\epsilon}\right|_{\epsilon} = 0 \quad .$$

Upon performing the substitution it is found that

$$\left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} = \int_{-\infty}^{\infty} \left\{ N(f)\, [H_o(f)\, H_2^*(f) + H_o^*(f)\, H_2(f)] - X(f)\, H(f)\, H_2(f) \sum_{k} 2\beta_k\, e^{j\omega k\mathcal{T}} \right\} df \quad . \tag{109}$$

However, since real-time functions are assumed and N(f) is a power spectral density,

$$\int_{-\infty}^{\infty} N(f)\, H_o(f)\, H_2^*(f)\, df = \int_{-\infty}^{\infty} N(-f)\, H_o(-f)\, H_2^*(-f)\, df = \int_{-\infty}^{\infty} N(f)\, H_o^*(f)\, H_2(f)\, df \quad .$$

Thus, Eq. (109) can be written

$$\left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} = 2 \int_{-\infty}^{\infty} H_2(f) \left[N(f)\, H_o^*(f) - X(f)\, H(f) \sum_{k} \beta_k\, e^{j\omega k\mathcal{T}} \right] df \quad . \tag{110}$$

Upon requiring that

$$\left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} = 0$$

for all $H_2(f)$ it follows that the bracketed quantity in the integrand of Eq. (110) must be zero for all f, i.e., $H_o(f)$ must be given by

$$H_o(f) = \frac{X^*(f)\, H^*(f)}{N(f)} \sum_{k=-N}^{N} \beta_k\, e^{-j\omega k\mathcal{T}} \quad . \tag{111}$$

It is interesting to note that $H_o(f)$ can be realized as the cascade of the optimum detector for colored noise followed by a "zero forcing" tapped delay line.

Next, it is necessary to solve for the $\{\beta_k\}$ that satisfy the intersymbol interference constraints. From Eq. (107) the output of $h_o(t)$ at $t = i\mathcal{T}$ is

† Observe that choosing $H_1(f)$ to <u>minimize</u> $\overline{[n_o(t)]^2}$ with z(0) fixed is equivalent to choosing $H_1(f)$ to maximize Eq. (108).

$$z(i\mathcal{T}) = \int_{-\infty}^{\infty} \frac{|X(f)\ H(f)|^2}{N(f)} \sum_k \beta_k \exp[j\omega(i-k)\mathcal{T}]\, df$$

$$= \sum_k \beta_k W_{ik} \tag{112}$$

where

$$W_{ik} \triangleq \int_{-\infty}^{\infty} \frac{|X(f)\ H(f)|^2}{N(f)} \exp[j\omega(i-k)\mathcal{T}]\, df \quad .$$

Thus, if the column vectors $\underline{z}$ and $\underline{\beta}$ are defined as

$$z \triangleq \begin{Bmatrix} z(-N\mathcal{T}) \\ \cdot \\ \cdot \\ \cdot \\ z(0) \\ \cdot \\ \cdot \\ \cdot \\ z(N\mathcal{T}) \end{Bmatrix} \qquad \beta \triangleq \begin{Bmatrix} \beta_{-N} \\ \cdot \\ \cdot \\ \cdot \\ \beta_o \\ \cdot \\ \cdot \\ \cdot \\ \beta_N \end{Bmatrix}$$

and if a matrix [W] is defined as

$$[W] \triangleq \begin{bmatrix} W_{-N,\,-N} & \cdots & W_{-N,\,0} & \cdots & W_{-N,\,N} \\ W_{0,\,-N} & \cdots & W_{0,\,0} & \cdots & W_{0,\,N} \\ W_{N,\,-N} & \cdots & W_{N,\,0} & \cdots & W_{N,\,N} \end{bmatrix}$$

it follows that

$$\underline{z} = [W]\,\underline{\beta} \quad . \tag{113}$$

Since [W] is known to be nonsingular,[93] the inverse matrix $[W]^{-1}$ exists and Eq. (113) becomes

$$[W]^{-1}\,\underline{z} = [W]^{-1}\,[W]\,\underline{\beta}$$

or

$$\underline{\beta} = [W]^{-1}\,\underline{z} \quad . \tag{114}$$

This is the desired expression that gives the tap gain settings β in terms of the constraint equations†

---

† Recall that in the maximization procedure z(0) was fixed but arbitrary. Since the value of z(0) has no effect on the SNR, the normalization z(0) = 1 has been assumed.

$$z(k\mathcal{T}) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases} \quad . \tag{115}$$

Finally, it is necessary to evaluate the SNR for this filter. From Eqs. (108) and (115) this is

$$\mathrm{SNR} = \left\{ \sum_{i,k} \beta_i \beta_k \int_{-\infty}^{\infty} \frac{|X(f)\ H(f)|^2}{N(f)} \exp\left[j\omega(i-k)\mathcal{T}\right] df \right\}^{-1}$$

or, from Eqs. (113) and (115)

$$\mathrm{SNR} = \beta_o^{-1} \quad . \tag{116}$$

For comparison purposes, it should be recalled[54] that the SNR at the output of the optimum detector for colored noise and an infinite observation interval is

$$\mathrm{SNR}\Big|_o = \int_{-\infty}^{\infty} \frac{|X(f)\ H(f)|^2}{N(f)} df \quad .$$

Thus, if X(f) is assumed normalized so that $\mathrm{SNR}\big|_o = 1$, β_o^{-1} will be the ratio of the SNR for the zero forcing filter to that for the optimum detector. Since this interpretation of β_o^{-1} is quite useful, the following examples assume X(f) to be so normalized.

Ideally, the performance of this modulation technique would now be evaluated by comparing the value of C for this approach with that obtained for the optimum technique. Unfortunately, when X(f), H(f), and N(f) are chosen so that the determination of β_o from Eq. (114) is feasible, it is impractical to determine C for the optimum technique. Conversely, when H(f) and N(f) are chosen to simplify evaluation of C for the optimum technique, it is impractical to evaluate β_o from Eq. (114). Thus, it is necessary to consider an alternate and simpler evaluation of the zero forcing technique. A relatively simple approach is to choose X(f), H(f), and N(f) so that β_o can be readily determined from Eq. (114). Then since β_o^{-1} represents the SNR degradation caused by the elimination of intersymbol interference, it seems reasonable to conclude that if for a given situation β_o is close to unity the performance of this technique would be acceptable. Conversely, if a situation is found in which β_o^{-1} is quite small, this technique would be unacceptable.

Recall from Eq. (111) that the zero forcing filter can be realized as the cascade of the optimum detector and a zero forcing tapped delay line. Viewed in this manner, it appears that the zero forcing procedure should lead to an excessive SNR loss only if the intersymbol interference at the detector output is in some sense large. The following examples confirm this speculation.

Let W(t) be the output of the optimum detector when x(t) is transmitted, i.e.,

$$W(t) = \int_{-\infty}^{\infty} \frac{|X(f)\ H(f)|^2}{N(f)} e^{j\omega t} df$$

and consider first the situation in which X(f), H(f), and N(f) are chosen so that W(t) appears as in Fig. 12. Then the elements in the [W] matrix are

$$W_{ik} = \begin{cases} 1 & i = k \\ 1/2 & i = k \pm 1 \\ 0 & \text{otherwise} \end{cases} \quad .$$

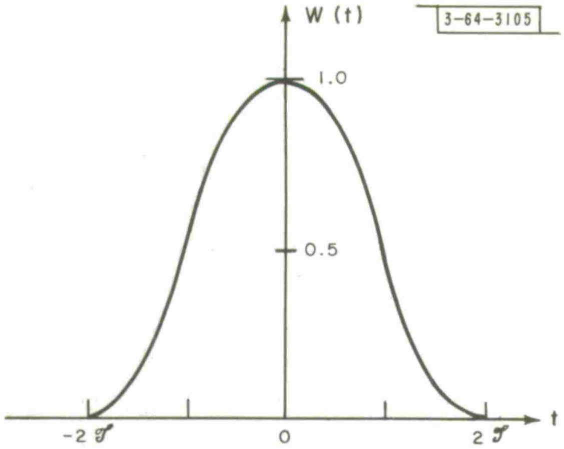


Fig. 12. Example in which intersymbol interference is large.

Calculations for N = 1, 2, 3 suggest that for this [W] the $\{\beta_k\}$ are given by

$$\beta_k = (-1)^{|k|}\,[N + 1 - |k|] \quad . \tag{117}$$

Substitution of Eq. (117) into Eq. (113) shows that

$$z(i\mathcal{T}) = \sum_{k=-N}^{N} \beta_k W_{ik} = (-1)^{|i|}\,[|i| - \tfrac{1}{2}\,|i-1| - \tfrac{1}{2}\,|i+1|] \qquad |i| \leqslant N$$

$$= \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases} \qquad |i| \leqslant N$$

which proves that Eq. (117) is valid for arbitrary N. Observe from Eq. (117) that $|\beta_{\pm N}| = 1$. Thus, the intersymbol interference between $x(t)$ and $x[t \pm (N + 1)\mathcal{T}]$ is independent of N. As a result, it would be necessary to set $N = \infty$ before zero intersymbol interference would be obtained for all sampling instants. However, Eq. (117) shows that $\beta_o = N + 1$. Therefore, as $N \rightarrow \infty$, $\beta_o^{-1} \rightarrow 0$, i.e., the SNR loss caused by the elimination of intersymbol interference becomes arbitrarily large as the intersymbol interference is eliminated for an arbitrarily large number of sampling instants. Thus, for this example, zero forcing is completely unacceptable as a technique for eliminating intersymbol interference.

Another example that gives some additional insight into the zero forcing filter is the following. Let X(f), H(f), and N(f) be chosen so that W(t) appears as in Fig. 13. Then the elements of [W] are

$$W_{ik} = \begin{cases} 1 & i = k \\ 0 & i = k \pm 1 \\ -1/4 & i = k \pm 2 \\ 0 & \text{otherwise} \end{cases} \quad .$$

For this [W] it is readily verified that the $\{\beta_k\}$ given by the following relations satisfy Eqs. (113) and (115).

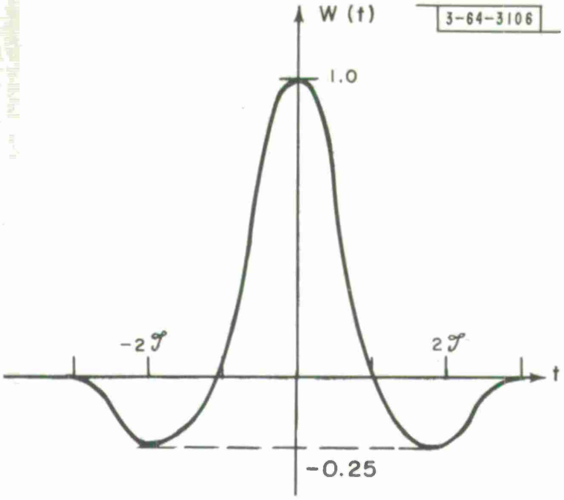


Fig. 13. Example in which intersymbol interference is small.

$$\beta_k = \beta_{-k}$$

$$\beta_k = 0, \text{ k odd}$$

$$\beta_N = 1/D$$

$$\beta_{N-2} = 4/D$$

$$\beta_{N-4} = 15/D$$

$$\vdots$$

$$\beta_k = 4\beta_{k+2} - \beta_{k+4} \qquad \text{for k even and } 0 \leqslant k \leqslant N-4 \tag{118}$$

D is determined from the relation $\beta_o - \frac{1}{2}\beta_2 = 1$. Thus, if N = 6, for example, the $\{\beta_k\}$ are

$$\beta_{\pm 1} = \beta_{\pm 3} = \beta_{\pm 5} = 0$$

$$\beta_o = 112/97, \ \beta_{\pm 2} = 30/97, \ \beta_{\pm 4} = 8/97, \ \beta_{\pm 6} = 1/97 \quad .$$

Two important results can be obtained from Eq. (118). First, observe that $\beta_{\pm N} = 1/D$. Although no general expression for D has been found, calculations show that D grows rapidly with N.† Thus, for only moderately large N, the intersymbol interference between x(t) and $x[t \pm (N+1)\mathcal{T}]$ will be negligible. This implies that from a practical standpoint, zero intersymbol interference can be obtained at all sampling instants by means of a finite length tapped delay line. This result, which is of considerable practical importance, should be contrasted with the analogous result in the previous example. Second, observe that β_o can be written as

$$\beta_o = \frac{1}{1 - \frac{1}{2}\frac{n_2}{n_0}}$$

† For example, the following results can be determined from Eq. (118).

N:	2	4	6	8	10
D:	7	15	48.5	181	675.5

where n_o and n_2 are the numerators of β_0 and β_2, respectively, i.e., $\beta_o = n_0/D$ and $\beta_2 = n_2/D$. Use of this result together with the iterative relations of Eq. (118) shows that, to slide rule accuracy,

$$\beta_o \simeq 1.15 \qquad \text{for all } N \quad .$$

Thus, the SNR loss due to the elimination of intersymbol interference is negligible and it may be concluded that for this example zero forcing is a satisfactory technique for eliminating intersymbol interference.

In summary, it appears from the previous examples that zero forcing is a suitable technique for eliminating intersymbol interference when the intersymbol interference at the optimum detector output is "suitably small," i.e., when the output has a ringing form as suggested by Fig. 13. Conversely, when the intersymbol interference is "large" as suggested by Fig. 12, the zero forcing procedure causes an excessive loss in SNR.

C. SUBSTITUTION OF SINUSOIDS FOR EIGENFUNCTIONS

The previous sections have considered two approaches to the elimination of intersymbol interference in suboptimum modulation systems. As indicated, the first of these approaches appears practical for relatively simple channels (an n = 1 Butterworth filter and white noise) while the second appears practical for "almost flat and band-limited" channels, i.e., channels for which the intersymbol interference at the optimum detector output is "relatively small." This section considers a third technique that appears practical for the range of channels between these extremes.

Recall that a study of suboptimum modulation systems involves finding "suitable substitutes" for the eigenfunctions of Theorems 1 to 3. The present section considers the possibility that suitable substitutes are time translates of a set of sinusoids. The motivation for this approach is provided in the following discussion.

1. Asymptotic Form of Eigenfunctions and Eigenvalues

This section is concerned with an investigation of the form of the $\{\varphi_i(t)\}$ and $\{\lambda_i\}$ of Theorems 1 to 3 for $T \to \infty$. Consider first the $\{\varphi_i(t)\}$ of Theorem 1 and recall that they satisfy the integral equation

$$\lambda_i \varphi_i(t) = \int_o^T \varphi_i(s)\, K(t,s)\, ds \qquad 0 \leqslant t \leqslant T \tag{119}$$

where

$$K(t,s) \triangleq \int_o^{T_1} h(\sigma - t)\, h(\sigma - s)\, d\sigma \quad .$$

Since the interval over which the $\{\varphi_i(t)\}$ are defined has no effect on the form of the $\{\varphi_i(t)\}$, it is convenient to consider the following equation instead of Eq. (119)

$$\lambda_i \varphi_i(t) = \int_{-T/2}^{T/2} \varphi_i(s)\, K(t,s)\, ds \qquad |t| \leqslant T/2 \tag{120}$$

where now

$$K(t, s) \triangleq \int_{-T/2}^{T_1-T/2} h(\sigma - t)\, h(\sigma - s)\, d\sigma$$

$$= \int_{-T'/2}^{T'/2} h(\sigma - t)\, h(\sigma - s)\, d\sigma$$

with $T' \triangleq T_1 - T/2$. The last equality follows from the fact that h(t) is physically realizable and $T_1 \geqslant T$ is assumed. Now, let $\varphi_i(t)$ be represented by a Fourier series, i.e., let

$$\varphi_i(t) = \sum_{k=-\infty}^{\infty} a_{ik}\, e^{j\omega_k t} \qquad \omega_k = \frac{2\pi k}{T} \qquad |t| \leqslant T/2 \quad . \tag{121}$$

Substitution of this expression into Eq. (120) shows that

$$\lambda_i \sum_k a_{ik}\, e^{j\omega_k t} = \sum_k a_{ik} \int_{-T/2}^{T/2} e^{j\omega_k s}\, K(t, s)\, ds \quad .$$

Multiplication of both sides by $\exp[-j\omega_\ell t]$ and integration over $[-T/2, T/2]$ gives

$$\lambda_i a_{i\ell} = \frac{1}{T} \sum_k a_{ik} \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} e^{j\omega_k s}\, K(t, s)\, e^{-j\omega_\ell t}\, ds\, dt \tag{122}$$

or, from Eq. (120)

$$\lambda_i a_{i\ell} = \frac{1}{T} \sum_k a_{ik} \int_{-T'/2}^{T'/2} \left[\int_{-T/2}^{T/2} h(\sigma - s)\, e^{j\omega_k s}\, ds\right] \left[\int_{-T/2}^{T/2} h(\sigma - t)\, e^{-j\omega_\ell t}\, dt\right] d\sigma \quad .$$

Since the bracketed terms are simply the output of the filter h(t) when the input is of the form $\exp[j\omega t]$, $-T/2 < t < T/2$, and since

$$\int_{-T/2}^{T/2} e^{j\omega_k t}\, e^{-j\omega t}\, dt = T\, \frac{\sin \pi(f - f_k)\, T}{\pi(f - f_k)\, T} \triangleq T\, \mathrm{sinc}\,(f - f_k)\, T$$

it follows that

$$\lambda_k a_{i\ell} = \frac{1}{T} \sum_k a_{ik} \int_{-T'/2}^{T'/2} \left[T \int_{-\infty}^{\infty} H(f)\, \mathrm{sinc}\,(f - f_k)\, T\, e^{j2\pi f\sigma}\, df\right]$$

$$\times \left[T \int_{-\infty}^{\infty} H(\nu)\, \mathrm{sinc}\,(\nu + f_\ell)\, T\, e^{j2\pi\nu\sigma}\, d\nu\right] d\sigma$$

or

$$\lambda_i a_{i\ell} = TT' \sum_k a_{ik} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(f)\, H(\nu)\, \mathrm{sinc}\,(\nu + f_\ell)\, T\, \mathrm{sinc}\,(f - f_k)\, T\, \mathrm{sinc}\,(f + \nu)\, T'\, df\, d\nu \quad . \tag{123}$$

Consider the integration over f and observe that unless $\nu \simeq -f_k$ the integral is approximately zero. Furthermore, if T' is assumed to be large enough so that H(f) is constant in both

amplitude <u>and</u> phase over a band of frequencies several times $1/T'$ cps wide about $f = -\nu$, it follows that when $\nu \simeq -f_k$, Eq. (123) can be written†

$$\lambda_i a_{i\ell} = T \sum_k a_{ik} \int_{-\infty}^{\infty} |H(\nu)|^2 \operatorname{sinc}(\nu + f_\ell) T \int_{-\infty}^{\infty} T' \operatorname{sinc}(f - f_k) T \operatorname{sinc}(f + \nu) T' \, df \, d\nu \quad . \tag{124}$$

Since $T' \geqslant T$,

$$\int_{-\infty}^{\infty} T' \operatorname{sinc}(f - f_k) T \operatorname{sinc}(f + \nu) T' \, df = \operatorname{sinc}(f_k + \nu) T$$

and Eq. (124) becomes

$$\lambda_i a_{i\ell} = T \sum_k a_{ik} \int_{-\infty}^{\infty} |H(\nu)|^2 \operatorname{sinc}(\nu + f_\ell) T \operatorname{sinc}(\nu + f_k) T \, d\nu \quad . \tag{125}$$

If the additional assumption is now made that T is large enough so that $|H(\nu)|^2$ is essentially constant over a band of frequencies several times $1/T$ cps wide centered about $\nu = -f_\ell$, Eq. (125) can be written

$$\begin{aligned} \lambda_i a_{i\ell} &= \sum_k a_{ik} |H(f_\ell)|^2 \int_{-\infty}^{\infty} T \operatorname{sinc}(\nu + f_\ell) T \operatorname{sinc}(\nu + f_k) T \, d\nu \\ &= a_{i\ell} |H(f_\ell)|^2 \end{aligned}$$

where the last step follows from the known orthogonality of sinc functions.[94] From this it follows that either $\lambda_i = |H(f_\ell)|^2$ or $a_{i\ell} = 0$, i.e., both $\exp[j\omega_\ell t]$ and $\exp[-j\omega_\ell t]$ (or equivalently $\sin \omega_\ell t$ and $\cos \omega_\ell t$) are eigenfunctions with eigenvalue $|H(f_\ell)|^2$. Thus, when T and T' are large enough to satisfy the assumed conditions, the $\{\varphi_i(t)\}$ and $\{\lambda_i\}$ of Theorem 1 are, to a good approximation, given by‡

$$\left.\begin{aligned} \varphi_{2i-1}(t) &= \sqrt{\frac{2}{T}} \sin \frac{2\pi i t}{T} \\ \varphi_{2i}(t) &= \sqrt{\frac{2}{T}} \cos \frac{2\pi i t}{T} \end{aligned}\right. \qquad \begin{aligned} & i = 1, 2, \ldots \\ & |t| \leqslant T/2 \end{aligned} \tag{126}$$

and

$$\lambda_{2i} = \lambda_{2i-1} = \left|H\left(\frac{i}{T}\right)\right|^2 \quad .$$

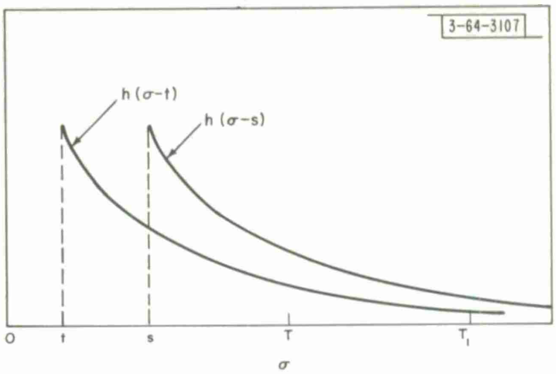
This is the desired asymptotic form for the eigenfunctions and eigenvalues of Theorem 1. Some additional insight into the value of T_1 required to make this result valid can be obtained by noting from Eq. (120) and Fig. 14 that if $T_1 > T + T_h$, where T_h is the "duration" of $h(t)$, to a good approximation, and for $-T/2 \leqslant t, s \leqslant T/2$,

---

† Note that in a strict sense, Eq. (124) is true only in the limit $T' \to \infty$. However, it is clear that when T' is sufficiently large, the error will be negligible.

‡ In this and subsequent discussions it is convenient to order the $\{\lambda_i\}$ in the manner of Eq. (126) rather than in the conventional manner of $\lambda_1 \geqslant \lambda_2 \geqslant \ldots$. It should be mentioned that other, closely related, asymptotic results have been obtained by Capon[95] and Rosenblatt[96] for the case of arbitrary T and $i \to \infty$.

Fig. 14. Concerning form of K(t,s) when T_1 is large.



$$K(t, s) = K(t - s)$$

where

$$K(t-s) = \int_{-\infty}^{\infty} |H(f)|^2 \exp[j\omega(t-s)]\, df \quad .$$

Substitution of this result into Eq. (122) leads directly to Eq. (125).

Consider next the $\{\varphi_i(t)\}$ and $\{\lambda_i\}$ of Theorem 2 and recall that

$$\lambda_i \varphi_i(t) = \int_0^T \varphi_i(\tau)\, K(t-\tau)\, d\tau \qquad 0 \leqslant t \leqslant T \tag{127}$$

where

$$K(t) \triangleq \int_{-\infty}^{\infty} \frac{|H(f)|^2}{N(f)} e^{j\omega t}\, df \quad .$$

As before, it is convenient to shift the time origin so that the $\{\varphi_i(t)\}$ are defined over $[-T/2, T/2]$ and therefore satisfy the equation

$$\lambda_i \varphi_i(t) = \int_{-T/2}^{T/2} \varphi_i(s)\, K(t-s)\, ds \qquad |t| \leqslant T/2 \quad . \tag{128}$$

Following the previous procedure, let $\varphi_i(t)$ be written as

$$\varphi_i(t) = \sum_{k=-\infty}^{\infty} a_{ik}\, e^{j\omega_k t} \qquad |t| \leqslant T/2$$

and substitute this expression into Eq. (128). This gives, after multiplication by $\exp[-j\omega \ell t]$ and integration over $[-T/2, T/2]$,

$$\lambda_i a_{i\ell} = \frac{1}{T} \sum_k a_{ik} \int_{-T/2}^{T/2} e^{-j\omega_\ell t} \left[\int_{-T/2}^{T/2} K(t-s)\, e^{j\omega_k s}\, ds \right] dt \quad .$$

Since the bracketed term is simply the output of the filter K(t) when the input is $\exp[j\omega_k t]$, $-T/2 < t < T/2$, and since the Fourier transform of this input is $T\ \mathrm{sinc}\,(f - f_k)\ T$, it follows that

$$\lambda_i a_{i\ell} = \sum_k a_{ik} \int_{-\infty}^{\infty} \frac{|H(f)|^2}{N(f)} \operatorname{sinc}(f - f_k) T \, df \int_{-T/2}^{T/2} \exp[-j(\omega + \omega_\ell) t] \, dt$$

$$= T \sum_k a_{ik} \int_{-\infty}^{\infty} \frac{|H(f)|^2}{N(f)} \operatorname{sinc}(f - f_k) T \operatorname{sinc}(f + f_\ell) T \, df \quad .$$

Assume now that T is large enough so that $|H(f)|^2/N(f)$ is essentially constant about $f = -f_\ell$. Then, with negligible error,

$$\lambda_i a_{i\ell} = \sum_k a_{ik} \frac{|H(f_\ell)|^2}{N(f_\ell)} \int_{-\infty}^{\infty} T \operatorname{sinc}(f - f_\ell) T \operatorname{sinc}(f + f_\ell) T \, df$$

$$= a_{i\ell} \frac{|H(f_\ell)|^2}{N(f_\ell)} \quad .$$

Thus, by analogy with the discussion preceding Eq. (126), it follows that when T satisfies the assumed condition, the $\{\varphi_i(t)\}$ and $\{\lambda_i\}$ of Theorem 2 are, with negligible error, given by

$$\varphi_{2i-1}(t) = \sqrt{\frac{2}{T}} \sin \frac{2\pi i t}{T} \qquad i = 1, 2, 3, \dots$$

$$\varphi_{2i}(t) = \sqrt{\frac{2}{T}} \cos \frac{2\pi i t}{T} \qquad |t| \leqslant T/2$$

$$\lambda_{2i} = \lambda_{2i-1} = \frac{|H(\frac{i}{T})|^2}{N(\frac{i}{T})} \tag{129}$$

which is the desired asymptotic form. At this point, it is sufficient to state that arguments similar to those used above show that for T and T_1 sufficiently large, the $\{\varphi_i(t)\}$ and $\{\lambda_i\}$ of Theorem 3 are also given by Eq. (129). From this the important conclusion is reached that in all cases the $\{\varphi_i(t)\}$ become sinusoids when $T \to \infty$. Furthermore, some idea of the value of T required to make this approximation valid has been obtained. This result together with the resulting simple form of the eigenvalues will prove useful in deriving a "good" suboptimum modulation system. Before proceeding with such a derivation, however, it is important to obtain some insight into the manner in which the $\varphi_i(t)$ differ from sinusoids when T is finite; this difference, although quite small, has been found to be important in an experimental system.

For this discussion, let T_1 be infinite, consider the $\{\varphi_i(t)\}$ of Theorem 1 and recall from Sec. II-D-4 that the $\{\varphi_i(t)\}$ are self-reproducing [over the interval $(0, T)$] when passed through the cascade of $h(t)$ and $h(-t)$, i.e., when passed through a filter whose impulse response is the autocorrelation function $R_h(t)$ of the channel filter. Next, observe as suggested in Fig. 15, that when T is long relative to the duration of $R_h(t)$ and a sinusoid of length T is passed through $R_h(t)$, the output signal differs from a sinusoid only near the ends of the interval. Thus, since the previous work has shown that for large T the $\{\varphi_i(t)\}$ are approximately sinusoids, and since sinusoids are self-reproducing except near the ends of the interval, it follows that for large but finite T the $\{\varphi_i(t)\}$ differ from sinusoids only near the ends of the interval.

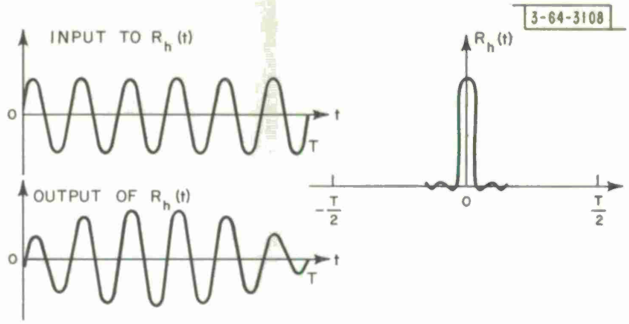


Fig. 15. Concerning form of the $\{\phi_i(t)\}$ when T is large.

2. Modulation System

This section is concerned with the derivation of a third suboptimum modulation system based on the results of the previous section and those in Appendix B.

Consider first the analysis of Appendix B and, for a given $|H(f)|^2/N(f)$, let T_m be chosen so that $|H(f)|^2/N(f)$ is essentially constant over an interval of $1/T_m$ cps. Then, from the discussion preceding Eq. (B-5), it is clear that for $T > T_m$ the sums in the error exponents for finite T will differ negligibly from the limiting integral forms. Next, recall from Sec. IV-C-1 that for $T > T_m$ the $\{\varphi_i(t)\}$ are, to a good approximation, sinusoids. Thus, since T might typically be on the order of days, weeks, or months and since, at least for telephone channels, T_m is on the order of 0.01 to 0.1 second, it seems reasonable to attempt to substitute for the $\{\varphi_i(t)\}$ time translates of a set of sinusoids approximately T_m seconds long. In other words, if a set of $\{\alpha_i(t)\}$ are defined as

$$\alpha_{2i-1}(t) \triangleq \begin{cases} \sqrt{\frac{2}{\mathcal{T}}}\,\sin\left[\frac{2\pi i t}{\mathcal{T}}\right] & |t| < \mathcal{T}/2 \\ 0 & \text{elsewhere} \end{cases} \qquad \mathcal{T} \approx T_m$$

$$\alpha_{2i}(t) = \begin{cases} \sqrt{\frac{2}{\mathcal{T}}}\,\cos\left[\frac{2\pi i t}{\mathcal{T}}\right] & |t| < \mathcal{T}/2 \\ 0 & \text{elsewhere} \end{cases} \tag{130}$$

the $\{\varphi_i(t)\}$ would be replaced by time translates (by $k\mathcal{T}$ seconds, k an integer) of some number of the $\{\alpha_i(t)\}$. However, when this is done two forms of intersymbol interference are encountered. One form occurs because the $\{\alpha_i(t)\}$ are only approximations to a set of $\{\varphi_i(t)\}$ of length $\mathcal{T}$ and therefore are only "approximately" orthogonal at the channel output. The second form arises from the nonorthogonality at the channel output of nonequal time translates of any two of the $\{\alpha_i(t)\}$. Both forms of intersymbol interference can be reduced significantly in the following manner.

Let white noise and an infinite observation interval be assumed† and recall from Sec. II-D-4 that the "coordinate filters" for this situation may be realized as the cascade of a filter with

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† The assumption of white noise is made here only to simplify the subsequent discussion. The results obtained can be applied to the colored noise problem by simply replacing the filter with impulse response h(−t) by a filter with transfer function $H^*(f)/N(f)$. The assumption of an infinite observation interval is made because it is usually quite easy to make the interval long enough to be infinite for all practical purposes and because implementation problems are simplified when this is done.

impulse response h(−t) followed by a multiply and integrate operation. Also, observe that if $\mathcal{T}$ in Eq. (130) is considerably greater than the duration of the impulse response of the cascade of h(t) and h(−t) [if $\mathcal{T}$ is considerably greater than the duration of the filter autocorrelation function $R_h(t)$] when $\alpha_i(t)$ is transmitted the corresponding signal at the output of h(−t) will be essentially an undistorted sinusoid except near the ends of the interval. Thus, if $\mathcal{T}_g$ is the approximate duration of $R_h(t)$ and if the $\{\alpha_i(t)\}$ are redefined as

$$\alpha_{2i-1}(t) \triangleq \begin{cases} \sqrt{\dfrac{2}{\mathcal{T}-\mathcal{T}_g}} \sin\left[\dfrac{2\pi i t}{\mathcal{T}-\mathcal{T}_g}\right] & |t| < \mathcal{T}/2 \\ 0 & \text{elsewhere} \end{cases}$$

$$\alpha_{2i}(t) = \begin{cases} \sqrt{\dfrac{2}{\mathcal{T}-\mathcal{T}_g}} \cos\left[\dfrac{2\pi i t}{\mathcal{T}-\mathcal{T}_g}\right] & |t| < \mathcal{T}/2 \\ 0 & \text{elsewhere} \end{cases} \tag{131}$$

it follows that by observing the output of h(−t) only over the "inner interval" of $|t| \leqslant (\mathcal{T} - \mathcal{T}_g)/2$ the orthogonality of the $\{\alpha_i(t)\}$ at the channel output will be greatly improved. In addition, when time translates of any two of the $\{\alpha_i(t)\}$ are transmitted, this same technique gives a significant reduction in the intersymbol interference between successive signals. The following example provides some quantitative insight into the effectiveness of this procedure. [Note that if $R_h(t)$ were identically zero for $|t| > \mathcal{T}_g$, <u>all</u> intersymbol interference would be completely eliminated.]

Consider a channel for which

$$h(t) = \begin{cases} e^{-t} & t \geqslant 0 \\ 0 & t < 0 \end{cases}$$

and let I_{ik} denote the magnitude of the number obtained when the output of h(−t), assuming $\alpha_i(t)$ is transmitted, is multiplied by $\alpha_k(t)$ and integrated over the interval $|t| \leqslant (\mathcal{T} - \mathcal{T}_g)/2$. (Observe that for $i \neq k$, I_{ik} provides a measure of the intersymbol interference caused by the nonorthogonality of the $\{\alpha_i(t)\}$ at the channel output.) For this channel and for $i \neq k$, I_{ik} can be upper bounded by

$$I_{ik} \leqslant \frac{2}{(\mathcal{T} - \mathcal{T}_g)} e^{-\mathcal{T}_g/2} \quad .$$

Thus, the intersymbol interference decreases almost exponentially with increasing $\mathcal{T}_g$ and is inversely proportional to the duration $\mathcal{T}$ of the $\{\alpha_i(t)\}$. For this channel, reasonable values of $\mathcal{T}_g$ and $\mathcal{T}$ might be 8 and 80 seconds, respectively.† For these values,

$$I_{ik} \lesssim 5.2 \times 10^{-4}$$

and since $I_{ii} = [1 + \omega_i]^{-1} \lesssim 1$, it follows that the intersymbol interference can be considered negligible except for extremely high SNR conditions.

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† The selection of values for $\mathcal{T}$ and $\mathcal{T}_g$ involves a trade-off between equipment complexity, intersymbol interference reduction, and loss in effective signal power. This point is considered in more detail later for telephone channels. The values assumed here are of the same order of magnitude (after an appropriate bandwidth scale factor is introduced) as those selected for the telephone line modulation system.

As the next step in the evaluation of this modulation system, it is of interest to investigate the performance that might be expected when it is used with a telephone channel.† Yudkin[34] has found that for several different circuits the autocorrelation function of the channel impulse response is essentially confined to an interval of 1 msec, i.e., $R_h(t) \simeq 0$ for $|t| > 500\ \mu$sec. Thus, a value of $\mathcal{T}_g = 1$ msec should prove satisfactory for such a system. Given this value of $\mathcal{T}_g$, $\mathcal{T}$ should be chosen so that $|H(f)|^2$ is approximately constant over an interval of $1/\mathcal{T}$ cps and also so that $\mathcal{T} \gg \mathcal{T}_g$. However, from the standpoint of equipment complexity, $\mathcal{T}$ should be as small as possible. A reasonable compromise is $\mathcal{T} = 11$ msec. With these parameters, the $\{\alpha_i(t)\}$ are sinusoids spaced at 100-cps intervals throughout the telephone line passband. To proceed further, consider a telephone line having a nearly flat amplitude characteristic over a band of 2.5 kcps and assume a SNR of 30 db.‡ For this line, approximately 50 of the $\{\alpha_i(t)\}$ would be used. If the situation is considered in which the modulation system is used without coding, it would be practical to assign equal energy to each of the $\{\alpha_i(t)\}$ and to let each signal carry n binary digits of information. The number n would be made as large as possible without causing an excessive error probability due to low level noise and intersymbol interference. Assuming negligible intersymbol interference, the value of n can be determined in the following manner.

By assumption, the transmitted signal (for one interval of $\mathcal{T}$ seconds) is of the form

$$x(t) = \sum_i x_i \alpha_i(t) \qquad |t| < \mathcal{T}/2 \tag{132}$$

where the $\{x_i\}$ are statistically independent and can assume the equiprobable values

$$\frac{x_i}{k} = \pm 1, \pm 3, \ldots, \pm 2^n - 1 \quad . \tag{133}$$

The constant k is chosen to satisfy the input power constraint. From Eqs. (131) and (132) k must satisfy

$$\begin{aligned} S = E\left[\frac{1}{\mathcal{T}} \int_{-T/2}^{T/2} x^2(t)\, dt\right] &= \frac{1}{\mathcal{T}} \sum_{i,j} \overline{x_i x_j} \int_{-T/2}^{T/2} \alpha_i(t)\, \alpha_j(t)\, dt \\ &= \frac{1}{\mathcal{T}} \sum_i \overline{x_i^2} \int_{-T/2}^{T/2} \alpha_i^2(t)\, dt = \frac{1}{\mathcal{T}(\mathcal{T} - \mathcal{T}_g)} \sum_i \overline{x_i^2} \int_{-T/2}^{T/2} [1 + (-1)^i \cos 2\omega_i t]\, dt \\ &= \frac{1}{\mathcal{T} - \mathcal{T}_g} \sum_i \overline{x_i^2} \left[1 + (-1)^i \frac{\sin \omega_i \mathcal{T}}{\omega_i \mathcal{T}}\right] \\ &\simeq \frac{1}{\mathcal{T} - \mathcal{T}_g} \sum_i \overline{x_i^2} \quad . \end{aligned} \tag{134}$$

† Since it is possible to effectively eliminate phase crawl at a negligible cost in signal power, its effects are not considered in this discussion.

‡ In defining the SNR it is assumed that the actual input signal power is scaled to compensate for any flat loss on the line. For example, if a particular line has a minimum loss of 10 db at some midband frequency, the actual input power would be decreased by 10 db in determining the SNR. Also, the noise power is considered to be the average power of the line output signal when the input signal is removed. (For analytical purposes, it is assumed later that the channel noise is white and Gaussian with double-sided noise spectral density N_o and that the noise power has been measured at the output of a 2.5-kcps ideal filter.)

where the last step follows from the fact that the low-frequency cutoff of the telephone line requires $\omega_i \gtrsim 2\pi(400)$ and therefore $\omega_i \mathcal{T} \gtrsim 8.8\pi$. From Eq. (133) it follows that

$$\overline{x_i^2} = k^2 2^{-n+1} \sum_{\ell=1}^{2^{n-1}} (2\ell - 1)^2 \qquad \text{all } i$$

or, from Series 25 of Ref. 99,

$$\overline{x_i^2} = \frac{1}{3} k^2 [2^{2n} - 1] \qquad \text{all } i \quad .$$

Assuming that 50 of the $\{\alpha_i(t)\}$ are used, it follows from this and Eq. (134) that k must satisfy

$$S = k^2 \frac{5 \times 10^3}{3} [2^{2n} - 1] \quad . \tag{135}$$

Next, it is necessary to determine the signal and noise components at the receiver output when the transmitted signal is given by Eq. (132). Since the k^{th} "coordinate filter" for this system is realized as the cascade of the filter $h(-t)$ followed by multiplication by $\alpha_k(t)$ and integration over the interval $|t| \leqslant (\mathcal{T} - \mathcal{T}_g)/2$, it follows that the signal portion of the k^{th} filter output, say s_k, is

$$s_k = \int_{-(\mathcal{T}-\mathcal{T}_g)/2}^{(\mathcal{T}-\mathcal{T}_g)/2} \alpha_k(t) \left[\int_{-\mathcal{T}/2}^{\mathcal{T}/2} x(\tau)\, R_h(t-\tau)\, d\tau\right] dt \quad .$$

Upon substituting Eq. (132) and recalling that zero intersymbol interference is assumed, it follows that

$$s_k = |H(f_k)|^2 x_k \tag{136}$$

where $f_k \triangleq [k + 1/2(\mathcal{T} - \mathcal{T}_g)]$, if k is odd and $f_k \triangleq k/2(\mathcal{T} - \mathcal{T}_g)$ if k is even.

Similarly, the noise portion of the k^{th} filter output, say n_k, is

$$n_k = \int_{-(\mathcal{T}-\mathcal{T}_g)/2}^{(\mathcal{T}-\mathcal{T}_g)/2} \alpha_k(t) \left[\int_{-\infty}^{\infty} n(\tau)\, h(\tau - t)\, d\tau\right] dt$$

and since $\overline{n(t)\, n(\tau)} = N_o \delta(t - \tau)$, it follows that

$$\overline{n_k n_j} = N_o \int_{-(\mathcal{T}-\mathcal{T}_g)/2}^{(\mathcal{T}-\mathcal{T}_g)/2} \alpha_k(t) \int_{-(\mathcal{T}-\mathcal{T})/2}^{(\mathcal{T}-\mathcal{T}_g)/2} \alpha_j(\tau)\, R_h(t-\tau)\, d\tau\, dt$$

$$\simeq N_o\, |H(f_k)|^2\, \delta_{kj} \tag{137}$$

where the last step follows from the assumption of zero intersymbol interference. Thus, if $|H(f_k)|^2 \simeq 1$ is assumed, it follows from Eq. (133), Eqs. (135) to (137), and the assumption of Gaussian noise that the probability of error is given by

$$P_e = 1 - \frac{1}{\sqrt{2\pi}\sigma} \int_{-\mu}^{\mu} \exp\left[-\frac{1}{2}\left(\frac{t}{\sigma}\right)^2\right] dt$$

$$= 1 - \frac{1}{\sqrt{2\pi}} \int_{-\mu/\sigma}^{\mu/\sigma} \exp\left[-\frac{1}{2} t^2\right] dt$$

where

$$\frac{\mu}{\sigma} = \left[\frac{3 \times 10^{-3} S}{5(2^{2n} - 1)}\right]^{1/2} [N_o]^{-1/2}$$

$$= \left[\frac{S}{N_o} \frac{3 \times 10^{-3}}{5} (2^{2n} - 1)^{-1}\right]^{1/2} \quad . \tag{138}$$

Finally, since a SNR of 30 db is assumed and since the SNR definition is

$$\mathrm{SNR} = \frac{S}{5N_o} \times 10^{-3}$$

it follows that

$$\frac{S}{5N_o} \times 10^{-3} = 10^3$$

or

$$\frac{S}{N_o} = 5 \times 10^6 \quad .$$

Thus, from Eq. (138),

$$\frac{\mu}{\sigma} = [3 \times 10^3 (2^{2n} - 1)^{-1}]^{1/2}$$

and it follows from a table of the Gaussian distribution that if $P_e < 10^{-5}$ is required, n must be such that

$$[3 \times 10^3 (2^{2n} - 1)^{-1}]^{1/2} > 4.42$$

or

$$n < \frac{1}{2} \log_2 155 \quad .$$

Thus, $n \leqslant 3$ and the conclusion is reached that it should be possible to transmit three binary digits of information on each of the $\{\alpha_i(t)\}$ without exceeding a P_e of 10^{-5}. Since the $\{\alpha_i(t)\}$ are 11 msec long and fifty of them are used, this implies a data rate of

$$R = 3 \times 50 \times (0.011)^{-1} = 13{,}600 \text{ bits/second} \quad . \tag{139}$$

When this result is compared with the rates of 2500 to 5000 bits/second attainable with commercial equipment (see Sec. I-C), it is clear that the present modulation system offers a significant potential improvement in performance. Furthermore, use of a "powerful" coder-decoder such as the SECO machine[50] would lead to a 10 to 20 percent rate increase while giving virtually error-free transmission.

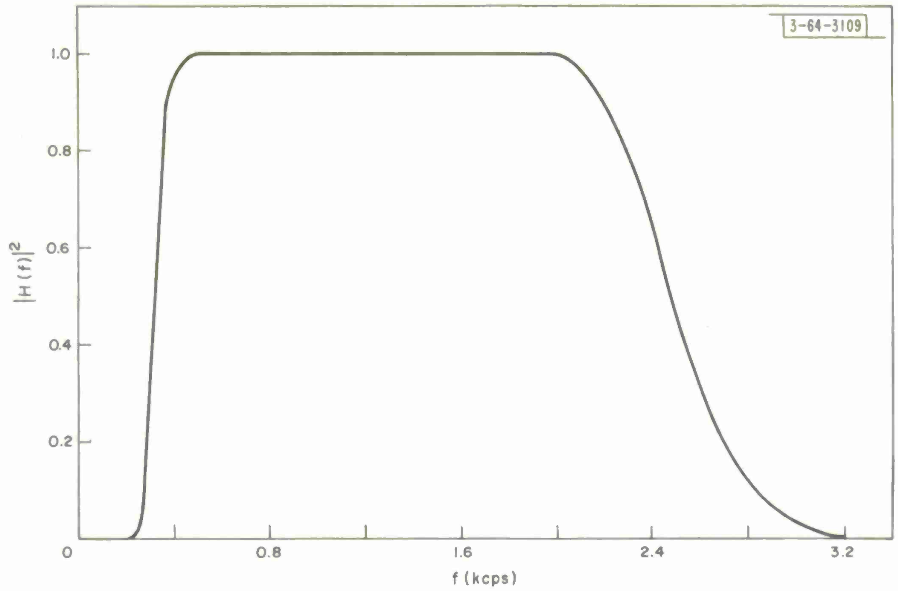

Fig. 16. Amplitude characteristic of simulated channel.

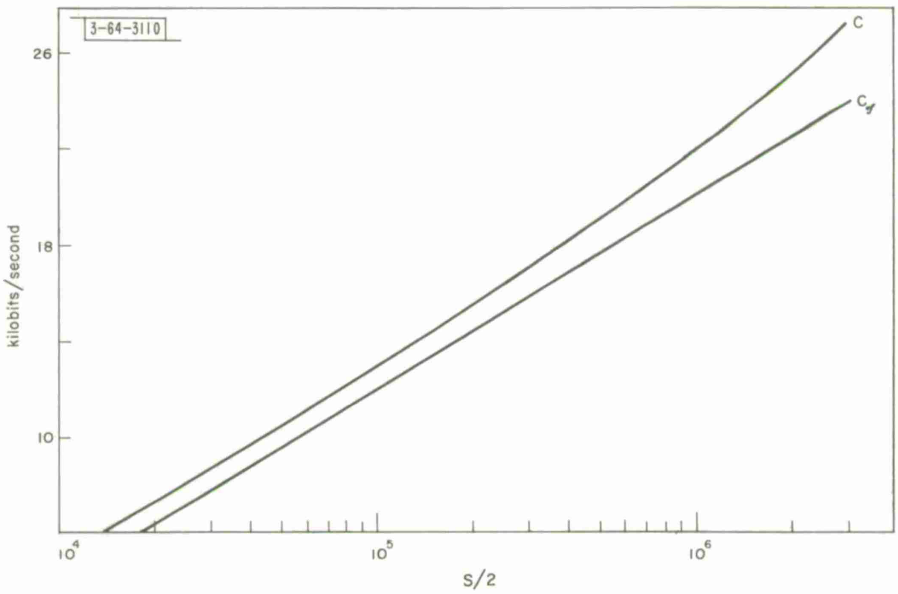

Fig. 17. C and $C_{\mathcal{J}}$ for channel in Fig. 16.

As a practical matter, however, it must be emphasized that this result is based on several assumed conditions that may be only approximately realized in practice. Most important of these is the assumption of negligible intersymbol interference. (Recall from Sec. I-D that intersymbol interference is the primary factor limiting the rate of present modems.) Since $R_h(t)$ is not identically zero for $|t| > 500\,\mu\text{sec}$, it is clear that some intersymbol interference must be present in an actual system; however, the form of $R_h(t)$ found by Yudkin[34] indicates that a significant reduction will be obtained. In view of the potential rate improvement with this modulation system and because of the impracticality of analytically determining the actual intersymbol interference level as well as the effects of other departures from assumed conditions, an experimental program was undertaken to demonstrate that the theoretical rate improvement could be realized in practice. The results of this program are presented in the following section.

As a final evaluation of the telephone line modulation system, it is of interest to compare the value of C for this modulation system, say $C_{\mathcal{T}}$, to the value obtained with the optimum modulation system for a channel whose amplitude characteristics are similar to those of a telephone line. Figure 16 presents the amplitude characteristic of a simulated channel used in the experimental work. Assuming white noise, the value of C for this channel is presented in Fig. 17 and has been obtained by graphical integration of Eq. (38). Assuming negligible intersymbol interference, it is readily found from the previous discussion and Eq. (35) that,

$$C_{\mathcal{T}} = \frac{1}{\mathcal{T}} \sum_{i\epsilon I} \log_2 \frac{\left|H\left(\frac{i}{\mathcal{T} - \mathcal{T}_g}\right)\right|^2}{B_{\mathcal{T}}(0)} \text{ bits/second} \tag{140}$$

where $B_{\mathcal{T}}(0)$ is chosen to satisfy

$$\frac{S}{2} = \frac{1}{\mathcal{T} - \mathcal{T}_g} \sum_{i\epsilon I} \left[\frac{1}{B_{\mathcal{T}}(0)} - \frac{1}{\left|H\left(\frac{i}{\mathcal{T} - \mathcal{T}_g}\right)\right|^2}\right]$$

and

$$I = \left[i: \left|H\left(\frac{i}{\mathcal{T} - \mathcal{T}_g}\right)\right|^2 > B_{\mathcal{T}}(0)\right] \quad .$$

Equation (140) is plotted in Fig. 17 for the channel of Fig. 16 using the values $\mathcal{T} = 11$ msec and $\mathcal{T}_g = 1$ msec. Observe that the effective loss in signal power is only 3 db at $S/2 = 3 \times 10^6$ which corresponds to a SNR of approximately 30 db.

CHAPTER V
EXPERIMENTAL PROGRAM

The experimental program described in this section was undertaken to demonstrate that the performance predicted for the telephone line modulation (TELMOD) system in Sec. IV-C-2 could be realized in practice. As described previously, the transmitter portion of the TELMOD system consists of approximately fifty different 11-msec duration sine and cosine signals spaced at 100-cps intervals throughout the telephone line passband. The receiver portion consists of filtering by $h(-t)$ followed by multiplication by the desired sine or cosine function and integration over the inner interval of 10 msec. In other words, with $\{\alpha_i(t)\}$ and $x(t)$ defined as in Eqs. (131) and (132), respectively, this system has the form indicated in Fig. 18 for the interval $|t| < \mathcal{T}/2$. Recall that the fundamental assumption of the previous analysis was that intersymbol interference for all signals was negligible relative to the assumed noise level of 30-db SNR. Thus, the primary goal of the experimental program was to demonstrate that this intersymbol interference level could be achieved in practice; a secondary goal, of course, was to demonstrate that other differences between the model and the real channel have a negligible effect on the predicted performance.

When considering the intersymbol interference level to be expected with the TELMOD system, two points should be noted. First, observe that if $\alpha_i(t)$ and $\alpha_j(t)$ differ in frequency by several hundred cycles per second, their spectra have virtually zero overlap. Thus, it would be expected that the orthogonality of these signals would be quite good at the channel output, i.e., the intersymbol interference at the output of a particular co-ordinate filter should be caused primarily by a few immediately adjacent tones. Second, recall from Sec. IV-C-1 that an eigenfunction is closely approximated by a sinusoid only if $|H(f)|^2$ is essentially constant over an interval several times $1/\mathcal{T}$ cps wide centered about the frequency of the sinusoid. Thus, it is to be expected that the orthogonality of immediately adjacent tones will be good (the intersymbol interference will be small) for tones in midband where $|H(f)|^2$ is nearly constant while the orthogonality will be poor near the band edges.

Based upon these considerations, an experiment was undertaken, which involved construction of transmitting and receiving equipment using ten of the $\{\alpha_i(t)\}$ discussed above with $\mathcal{T}$ = 11 msec and $\mathcal{T}_g$ = 1 msec. The amplitude of each $\alpha_i(t)$ is determined by a five-digit binary number; this number is obtained from either manual switches or a random source. The frequencies of the $\{\alpha_i(t)\}$ were variable and covered with ranges of 0.5 to 0.9 kcps, 1.5 to 1.9 kcps and 2.5 to 2.9 kcps with both sine and cosine signals being used at each frequency. The block diagram for this system is shown in Fig. 19.

Using this equipment, the following experiments were performed.

A. SIMULATED CHANNEL TESTS

On a simulated channel with white Gaussian noise and the filter amplitude characteristic presented in Fig. (16), the zero rate error exponent was evaluated by measuring the probability of error for two orthogonal signals. The measured performance was 0.8 to 1.0 db from theoretical which is well within the loss attributable to measurement inaccuracies and slight equipment imperfections. The intersymbol interference level for various tones was investigated by measuring the error rate due to intersymbol interference in the absence of noise. Using the

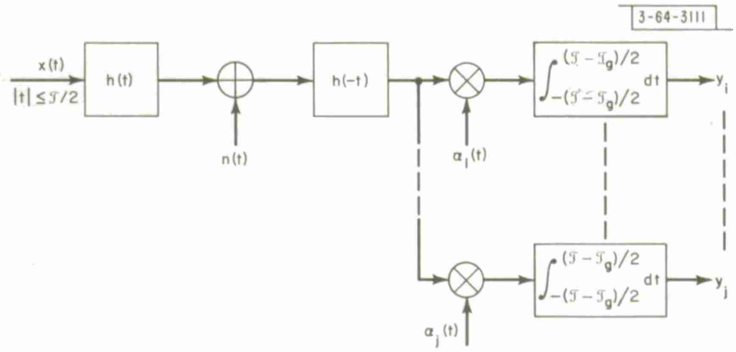

Fig. 18. Block diagram of TELMOD system.

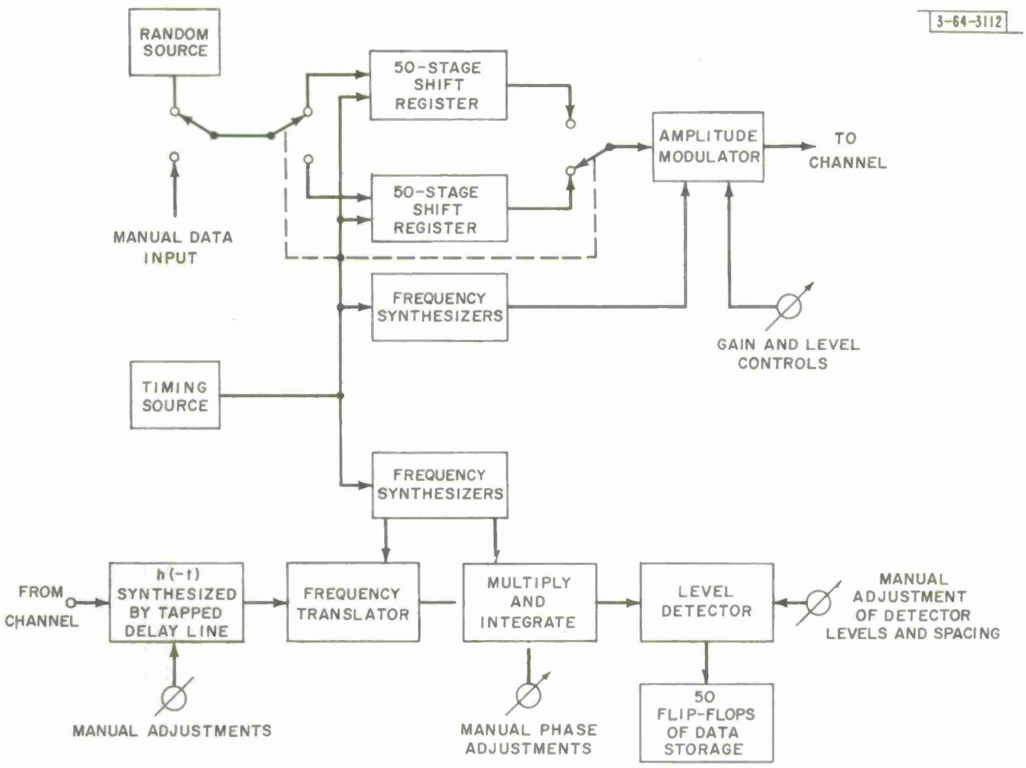

Fig. 19. Block diagram of experimental equipment.

approximation that the intersymbol interference was Gaussianly distributed† it was possible to calculate an effective intersymbol interference variance for various tones in the frequency bands of 0.5 to 0.9, 1.5 to 1.9 and 2.5 to 2.9 kcps. The results of these measurements can be summarized as follows.‡

(1) In the region of 0.6 to 2.0 kcps, where $|H(f)|^2$ is flat, the intersymbol interference variance between a particular reference tone and adjacent tones decreased approximately as $1/n$, where n is the frequency difference between the tones in units of 100 cps. For $n > 6$, the intersymbol interference variance caused by a single tone was too small to be measured.

(2) In the frequency range of 0.9 to 1.9 kcps, the total intersymbol interference variance§ was less than 6.25×10^{-2}. For 16-level output quantization, this implies that $P_e \lesssim 3 \times 10^{-5}$. Thus, to within the accuracy of the Gaussian approximation to the intersymbol interference distribution, it would be possible to transmit four binary digits of information on each of the tones.

(3) For frequencies in the range of 0.5 to 0.9 kcps, the total intersymbol interference variance increased from a level of less than 6.25×10^{-2} at 0.9 kcps, to 0.2 to 0.7 kcps, to 1.0 at 0.5 kcps. The corresponding information rates at a $P_e \lesssim 3 \times 10^{-5}$ would be 4, 3, and 2 binary digits per tone for the frequencies of 0.9, 0.7, and 0.5 kcps, respectively.

(4) In the 1.9- to 3.0-kcps frequency range, the intersymbol interference variance increased from 6.25×10^{-2} at 1.9 kcps to 0.16 at 2.5 kcps, to 0.5 at 2.7 kcps, to 1.25 at 2.9 kcps. The corresponding information rate at a $P_e \lesssim 3 \times 10^{-5}$ would be 3, 2, and 1 binary digits per tone for the frequencies of 2.5, 2.7, and 2.9 kcps, respectively.

The significance of the measurements on the simulated channel can be summarized as follows.

(1) As predicted theoretically, the intersymbol interference level is small for signals in the frequency range where $|H(f)|^2$ is constant (in the 0.9- to 1.9-kcps range) while it is significantly large near either band edge, where $|H(f)|^2$ changes considerably over an interval of a few hundred cycles per second.

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† The justification for this assumption is the fact that the intersymbol interference for a given tone is a sum of the (small) random interferences from a number of adjacent tones. Thus, to a first approximation it would be expected from the Central Limit Theorem that the total interference would be Gaussianly distributed. Clearly, the fact that only 10 to 12 tones contribute significantly to the intersymbol interference for a given tone implies that this is a highly approximate assumption. However, it is a convenient engineering approximation which led to consistent results.

‡ All the results presented in this section are based on an extrapolation of measurements made in the indicated frequency bands. For example, the intersymbol interference variance for a tone at 0.9 kcps was obtained by measuring the intersymbol interference variance due to the signals in the 0.5- to 0.8-kcps band and adding this to the variance obtained for the 0.9-kcps tone when <u>all</u> signals were transmitted with random amplitude.

§ The intersymbol interference variances given in this section are expressed in normalized form, the reference being taken as the amount by which an integrator output changes when the amplitude of the corresponding input signal is changed by one level. Thus, for a given intersymbol interference variance σ^2, the probability of error, assuming 32-level quantization of the integrator output, is

$$P_e = 1 - (2\pi\sigma^2)^{-1/2} \int_{-1/2}^{1/2} \exp\left[-\frac{1}{2}(x/\sigma)^2\right] dx \quad .$$

Similarly, if 16-level quantization is assumed

$$P_e = 1 - (2\pi\sigma^2)^{-1/2} \int_{-1}^{1} \exp\left[-\frac{1}{2}(x/\sigma)^2\right] dx \quad .$$

(2) On the basis of midband measurements, the intersymbol interference level for the TELMOD system on this channel is equivalent to a noise level of approximately 31-db SNR.† Thus, for a noise SNR less than about 30 db, the intersymbol interference level could be considered negligible, and furthermore, the performance would be within 1 db of theory. Conversely, for a noise SNR significantly greater than 30-db intersymbol interference would be the primary factor in determining probability of error.

(3) If a noise SNR considerably greater than 30 db is assumed and if a $P_e \lesssim 3 \times 10^{-5}$ is desired, it would be possible to achieve a data rate of approximately 14,000 bits/second over this channel. This number is based upon the measurements given above and assumes the following relation between the tones and the quantization levels.

Frequency (kcps)	Binary Digits/Tone
0.5 , 0.6	2
0.7 , 0.8	3
0.9 to 1.9	4
2.0 to 2.4	3
2.5 to 2.7	2
2.8 to 3.0	1

Furthermore, if the noise SNR were sufficiently good, it would be possible to do amplitude equalization prior to the h(−t) filter and thus to extend the flat portion of the channel to 3.0 kcps.‡ Such equalization would lead to a rate of 15,000 bits/second and would greatly simplify instrumentation problems.

B. DIAL-UP CIRCUIT TESTS

Intersymbol interference measurements were made on a local dial-up line whose amplitude characteristics are given in Fig. 20. For this channel, it was found that the severe amplitude ripple gave an unusually long autocorrelation function $R_h(t)$ of approximately 10-msec duration. Because of this, the observed intersymbol interference levels were large and the assumption of a Gaussian distribution for the intersymbol interference could not be made. Thus, the only measurements possible for this channel were probability of error vs number of amplitude levels. The results of these measurements indicate that only two levels per tone (or equivalently about 1800 bits/second) can be transmitted at an acceptable error rate using the TELMOD system presented in Fig. 18. However, it was found that the low-level noise SNR for this line was better than 50 db.§ Thus, amplitude equalization of the channel before filtering by h(−t) could be done

† From Eq. (138) and the subsequent discussion, it follows that if n is the number of binary digits that can be transmitted at a $P_e \lesssim 3 \times 10^{-5}$ due to intersymbol interference, the effective intersymbol interference SNR is given by

$$\left[\mathrm{SNR} \times \frac{3}{2^{2n} - 1}\right]^{1/2} \simeq 4.0 \quad .$$

‡ As a practical matter, it has been found during the experimental work that the SNR for many leased data circuits is on the order of 50 db rather than the frequently quoted 30 db. Thus, such a procedure of amplitude equalization, although not optimum from the standpoint of detection theory, would lead to significantly reduced intersymbol interference and thus to an increased rate.

§ It must be emphasized that this SNR was observed only in the absence of impulse noise. For this line, the impulse noise was both large and frequently occurring; the observed probability of error due only to impulse noise was on the order of 10^{-3}.

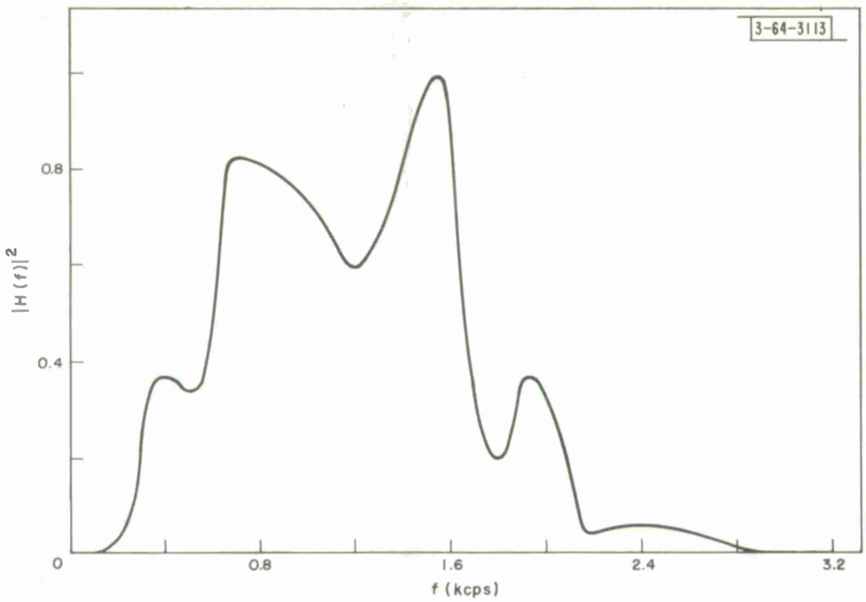

Fig. 20. Amplitude characteristic of local dial-up line.

to give a flat response over the band of approximately 0.6 to 1.6 kcps. On the basis of the tests on the simulated channel, this would give a capability of transmitting at least eight levels on each tone and thus of realizing a data rate of approximately 6000 bits/second.

C. SCHEDULE 4 DATA CIRCUIT TESTS

The third series of tests were made on a Schedule 4 Data Circuit looped from Lincoln Laboratory via Springfield, Massachusetts, whose amplitude characteristics are given in Fig. 21. As with the simulated channel, the intersymbol interference level for various tones was investigated by measuring probability of error due to intersymbol interference and then calculating an effective intersymbol interference variance assuming a Gaussian distribution.† The results of these measurements can be summarized as follows.

(1) As before, the intersymbol interference between a reference tone and adjacent tones was found to decrease approximately as $1/n$ where the frequency difference between tones is $n \times 100$ cps. For $n > 6$, the intersymbol interference due to a single tone was too small to be measured.

(2) In the frequency range of 0.5 to 0.9 kcps, the intersymbol interference variance was 0.74 at 0.5 kcps and 1.56 at 0.7 and 0.9 kcps.‡ The corresponding data rate at a $P_e \lesssim 3 \times 10^{-5}$ is 2 binary digits per tone at 0.5 kcps and 1 binary digit per tone at 0.7 and 0.9 kcps.

(3) For the frequency range 1.5 to 1.9 kcps, the measured variances were 0.12 at 1.5 kcps, 0.15 at 1.7 kcps and 0.21 at 1.9 kcps. The corresponding data rate would be 3 binary digits per tone.

† For this line, the low-level noise SNR was 50 db and impulse noise activity was extremely small; the measured probability of error due only to impulse noise was on the order of 10^{-6}. Thus, the effects of both low-level noise and impulse noise were neglected in the intersymbol interference measurements.

‡ Observe here that the intersymbol interference <u>increases</u> for tones further away from the band edge. The cause of this is the rapid change in $|H(f)|^2$ around 1.0 kcps.

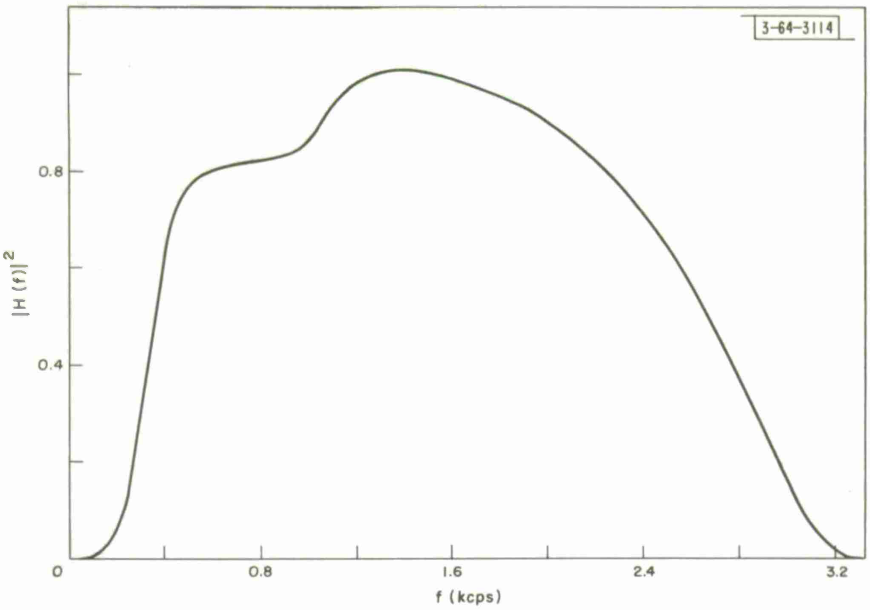

Fig. 21. Amplitude characteristic of Schedule 4 Data Circuit.

(4) In the 2.5- to 2.9-kcps frequency range, the variances increased from 1.32 at 2.5 kcps to 2.1 at 2.7 kcps, to 3.3 at 2.9 kcps. The corresponding data rate at a $P_e \lesssim 3 \times 10^{-5}$ is 1 binary digit per tone.

The significance of these measurements can be summarized as follows.

(1) As predicted theoretically, the fact that this channel is not flat causes the intersymbol interference level for the TELMOD system to be significantly higher than it was on the simulated channel. However, the high observed SNR for low-level noise would allow amplitude equalization of the channel prior to filtering by h(−t) and thus would allow a significant reduction in the intersymbol interference level.

(2) If a $P_e \lesssim 3 \times 10^{-5}$ is desired, it would be possible without amplitude equalization of the line to achieve a data rate of approximately 8400 bits/second. This is based upon the measurements given above and assumes the following relation between the tones and the quantization levels.

Frequency (kcps)	Binary Digits/Tone
0.5 to 1.0	1
1.1 to 1.3	2
1.4 to 1.9	3
2.0 to 2.4	2
2.5 to 3.0	1

If, however, amplitude equalization of the line was performed, the tests on the simulated channel indicate that a rate of approximately 15,000 bits/second would be possible.

APPENDIX A
PROOF THAT THE KERNELS OF THEOREMS 1 TO 3 ARE $\mathcal{L}_2$

A kernel K(t, s) is said to be an $\mathcal{L}_2$-kernel if[65]

$$||K||^2 \triangleq \int_0^T \int_0^T K^2(t, s)\, dt\, ds < \infty \quad .$$

For the kernel of Theorem 1, it follows that

$$||K||^2 = \int_0^T \int_0^T \left[\int_0^{T_1} h(\sigma - t)\, h(\sigma - s)\, d\sigma\right]^2 dt\, ds$$

and thus from the Schwarz inequality

$$||K||^2 \leqslant \int_0^T \int_0^T \left[\int_0^{T_1} h^2(\sigma - t)\, d\sigma\right] \left[\int_0^{T_1} h^2(\sigma - s)\, d\sigma\right] dt\, ds$$

$$= \left[\int_0^T \int_0^{T_1} h^2(\sigma - t)\, d\sigma\, dt\right]^2$$

$$\leqslant \left[\int_0^T \int_{-\infty}^{\infty} h^2(\sigma)\, d\sigma\, dt\right]^2$$

$$= \left[T \int_{-\infty}^{\infty} h^2(t)\, dt\right] < \infty$$

which proves that the kernel is $\mathcal{L}_2$. Next, to prove that the kernel of Theorem 2 is $\mathcal{L}_2$ define a function $K_{1/2}(t)$ by

$$K_{1/2}(t) \triangleq \int_{-\infty}^{\infty} |H(f)|\, [N(f)]^{-1/2}\, e^{j\omega t}\, df \quad .$$

Then

$$K(t - s) = \int_{-\infty}^{\infty} K_{1/2}(t - \sigma)\, K_{1/2}(s - \sigma)\, d\sigma$$

and, therefore

$$||K||^2 = \int_0^T \int_0^T \left[\int_{-\infty}^{\infty} K_{1/2}(t - \sigma)\, K_{1/2}(s - \sigma)\, d\sigma\right]^2 dt\, ds$$

which, from the Schwarz inequality, becomes

$$||K||^2 \leqslant \int_0^T \int_0^T \left[\int_{-\infty}^{\infty} K^2_{1/2}(t - \sigma)\, d\sigma\right] \left[\int_{-\infty}^{\infty} K^2_{1/2}(s - \sigma)\, d\sigma\right] dt\, ds$$

$$= \left[\int_0^T dt \int_{-\infty}^{\infty} K^2_{1/2}(\sigma)\, d\sigma\right]^2 \quad .$$

Thus, from Parseval's theorem,

$$||K||^2 \leqslant \left[T \int_{-\infty}^{\infty} |H(f)|^2 \, [N(f)]^{-1} \, df \right]^2 < \infty$$

and the kernel is $\mathfrak{L}_2$. Finally, to prove that the kernel of Theorem 3 is $\mathfrak{L}_2$ define a function $k_{1/2}(t, s)$ by

$$k_{1/2}(t, s) \triangleq \sum_{i=1}^{\infty} \frac{(h_t, \gamma_i)_{T_1}}{\sqrt{\beta_i}} \gamma_i(s)$$

where

$$(h_t, \gamma_i)_{T_1} \triangleq \int_0^{T_1} \gamma_i(\sigma) \, h(\sigma - t) \, d\sigma$$

and $\gamma_i(t)$ and β_i are defined as in the proof of Theorem 3. Then

$$\begin{aligned}
\int_0^{T_1} k_{1/2}(t, \sigma) \, k_{1/2}(s, \sigma) \, d\sigma &= \sum_{i, j} \frac{(h_t, \gamma_i)_{T_1} (h_s, \gamma_j)_{T_1}}{\sqrt{\beta_i \beta_j}} (\gamma_i, \gamma_j)_{T_1} \\
&= \sum_{i} \frac{(h_t, \gamma_i)_{T_1} (h_s, \gamma_i)_{T_1}}{\beta_i} \\
&= \int_0^{T_1} \int_0^{T_1} h(\sigma - t) \left[\sum_{i=1}^{\infty} \frac{\gamma_i(\sigma) \, \gamma_i(\rho)}{\beta_i} \right] h(\rho - s) \, d\sigma \, d\rho \\
&= K(t, s) \quad .
\end{aligned}$$

Thus,

$$||K||^2 = \int_0^T \int_0^T \left[\int_0^{T_1} k_{1/2}(t, \sigma) \, k_{1/2}(s, \sigma) \, d\sigma \right]^2 dt \, ds$$

which becomes, from the Schwarz inequality,

$$\begin{aligned}
||K||^2 &\leqslant \int_0^T \int_0^T \left[\int_0^{T_1} k_{1/2}^2(t, \sigma) \, d\sigma \right] \left[\int_0^{T_1} k_{1/2}^2(s, \sigma) \, d\sigma \right] dt \, ds \\
&= \left[\int_0^T \sum_{i=1}^{\infty} \frac{(h_t, \gamma_i)_{T_1}^2}{\beta_i} \, dt \right]^2 \quad .
\end{aligned}$$

However, it is known[98] that

$$\sum_{i=1}^{\infty} \frac{(h_t, \gamma_i)_{T_1}^2}{\beta_i} \leqslant \int_{-\infty}^{\infty} \frac{|H(f)|^2}{N(f)} \, df \quad .$$

Thus,

$$||K||^2 \leqslant \left[T \int_{-\infty}^{\infty} \frac{|H(f)|^2}{N(f)}\right]^2 < \infty$$

and the kernel is $\mathfrak{L}_2$.

APPENDIX B
DERIVATION OF ASYMPTOTIC FORM OF ERROR EXPONENTS

This Appendix is concerned with a derivation of the limiting form of the error exponents of Eqs. (34) to (36) and Eq. (58) for $T \to \infty$. Consider first the standard random coding exponent given by

$$E_T(\rho) = \left(\frac{\rho}{1+\rho}\right)^2 \frac{S}{2} B_T(\rho) \qquad R(1) \leqslant R \leqslant R(0)$$

$$R(\rho) = \frac{1}{2T} \sum_{i=1}^{N} \ln \frac{\lambda_i}{B_T(\rho)} - \frac{E_T(\rho)}{\rho} \qquad 0 \leqslant \rho \leqslant 1 \tag{B-1}$$

and

$$E_T(R) = \frac{1}{2T} \sum_{i=1}^{N} \ln \frac{\lambda_i}{B_T(1)} - R \qquad 0 \leqslant R \leqslant R(1)$$

where

$$\frac{1}{B_T(\rho)} = \frac{\frac{ST}{1+\rho} + \sum_{i=1}^{N} \lambda_i^{-1}}{N}$$

and N is chosen to satisfy $\lambda_N > B_T(\rho) \geqslant \lambda_{N+1}$. Recall from Sec. IV-C-1 that for large T the $\{\lambda_i\}$ are given by

$$\lambda_{2i} = \lambda_{2i-1} = \frac{|H(\frac{i}{T})|^2}{N(\frac{i}{T})} \qquad i = 1, 2, 3, \ldots \tag{B-2}$$

and assume temporarily that $|H(f)|^2/N(f)$ is a continuous monotone nonincreasing function for $f \geqslant 0$. Also, recall from Fig. 5(a), or observe from Eq. (B-1), that $B_T(\rho)$ is chosen to satisfy

$$\frac{S}{2(1+\rho)} = \frac{1}{T} \sum_{i=1}^{N/2} \left[\frac{1}{B_T(\rho)} - \frac{N(\frac{i}{T})}{|H(\frac{i}{T})|^2}\right] \tag{B-3}$$

where now $N/2$ is the largest integer such that

$$\frac{|H(\frac{N}{2T})|^2}{N(\frac{N}{2T})} > B_T(\rho) \quad .$$

Observe, as suggested in Fig. B-1, that the quantity

$$\frac{1}{T} \sum_{i=1}^{N/2} \frac{N(\frac{i}{T})}{|H(\frac{i}{T})|^2} \tag{B-4}$$

is an approximation to the area under the curve $N(f)/|H(f)|^2$, while the quantity

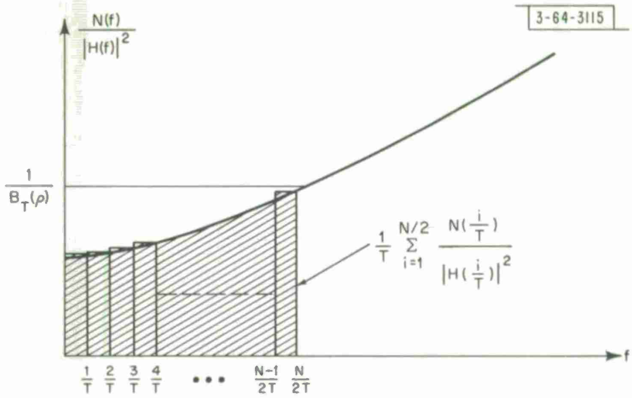

Fig. B-1. Concerning interpretation of Eq. (B-3).

$$\frac{N}{2T} \cdot \frac{1}{B_T(\rho)}$$

is precisely the area under the line $[B_T(\rho)]^{-1}$. Viewed in this manner, it is clear that for arbitrarily large T, $B_T(\rho)$ will always be adjusted to make the difference in these two areas equal to $S/2(1 + \rho)$. Thus, since $N(f)/|H(f)|^2$ is continuous and therefore Rieman integrable over a finite interval,[69] and since it is readily shown[97] from the definition of a Rieman integral that when $N = 2TW (W > 0)$

$$\lim_{T\to\infty} \frac{1}{T} \sum_{i=1}^{N/2} \frac{N(\frac{i}{T})}{|H(\frac{i}{T})|^2} = \int_0^W \frac{N(f)}{|H(f)|^2}\, df$$

it follows that in the limit $T \to \infty$, $B_T(\rho)$ or, more simply, $B(\rho)$ must be chosen to satisfy

$$\frac{S}{2(1+\rho)} = \int_0^W \left[\frac{1}{B(\rho)} - \frac{N(f)}{|H(f)|^2}\right] df \tag{B-5}$$

where W is defined by

$$\frac{|H(W)|^2}{N(W)} = B(\rho) \quad . \tag{B-6}$$

From this discussion and Eq. (B-1) it follows that

$$E(\rho) \triangleq \lim_{T\to\infty} E_T(\rho) = \left(\frac{\rho}{1+\rho}\right)^2 \frac{S}{2} B(\rho) \qquad 0 \leqslant \rho \leqslant 1 \tag{B-7}$$

where $B(\rho)$ is given by Eq. (B-6). Similarly, from Eqs. (B-1), (B-2), and (B-5) if follows that

$$\lim_{T\to\infty} R(\rho) = \lim_{T\to\infty} \left[\frac{1}{T} \sum_{i=1}^{N/2} \ln \frac{|H(i/T)|^2}{N(i/T)} - \frac{N}{2T} \ln B_T(\rho) - \frac{E_T(\rho)}{\rho}\right]$$

$$= \lim_{T\to\infty} \left[\frac{1}{T} \sum_{i=1}^{N/2} \ln \frac{|H(i/T)|^2}{N(i/T)} - \frac{N}{2T} \ln B(\rho)\right] - \frac{E(\rho)}{\rho} \qquad 0 \leqslant \rho \leqslant 1 \quad . \tag{B-8}$$

From Fig. B-1 it is clear that for $T \to \infty$, $N/2T \to W$ where W is defined by Eq. (B-6). Furthermore, since $\ln(\cdot)$ is a continuous function of its argument, it follows from above that

$$\lim_{T\to\infty} \frac{1}{T} \sum_{i=1}^{N/2} \ln \frac{|H(i/T)|^2}{N(i/T)} = \int_0^W \ln \frac{|H(f)|^2}{N(f)}\, df \quad .$$

Thus, from Eq. (B-8),

$$\lim_{T\to\infty} R(\rho) = \int_0^W \ln \frac{|H(f)|^2}{N(f)}\, df - W \ln B(\rho) - \frac{E(\rho)}{\rho}$$

$$= \int_0^W \ln \frac{|H(f)|^2}{N(f)\ B(\rho)}\, df - \frac{E(\rho)}{\rho} \qquad 0 \leqslant \rho \leqslant 1 \quad . \tag{B-9}$$

Equations (B-7) and (B-9) are the desired asymptotic forms when $|H(f)|^2/N(f)$ is monotonic. When $|H(f)|^2/N(f)$ is nonmonotonic an analogous derivation shows that for $T \to \infty$, $B(\rho)$ is chosen so that

$$\frac{S}{2(1+\rho)} = \int_W \left[\frac{1}{B(\rho)} - \frac{N(f)}{|H(f)|^2}\right] df \tag{B-10}$$

where

$$W = \left\{+f : \frac{|H(f)|^2}{N(f)} > B(\rho)\right\} \quad .$$

The corresponding values of $E(\rho)$ and $R(\rho)$ are given by

$$E(\rho) = \left(\frac{\rho}{1+\rho}\right)^2 \frac{S}{2} B(\rho) \qquad 0 \leqslant \rho \leqslant 1$$

and

$$R(\rho) = \int_W \ln \frac{|H(f)|^2}{N(f)\ B(\rho)}\, df - \frac{E(\rho)}{\rho} \qquad 0 \leqslant \rho \leqslant 1$$

which is the result presented in Eqs. (37) and (38).

When $0 \leqslant R \leqslant R(1)$, it follows from Eq. (B-1) and the previous discussion that

$$\lim_{T\to\infty} E_T(R) = \lim_{T\to\infty} \frac{1}{T} \sum_{i=1}^{N/2} \ln \frac{|H(i/T)|^2}{N(i/T)\ B_T(1)} - R$$

$$= \int_W \ln \frac{|H(f)|^2}{N(f)\ B(1)}\, df - R$$

where $B(1)$ and W are defined by Eq. (B-10). This is the result presented in Eq. (39).

Finally, the similarity of the expurgated bound given in Eq. (58) to the standard bound given in Eq. (B-1) together with the above results leads directly to the asymptotic form of the expurgated bound presented in Eq. (59).

APPENDIX C
DERIVATION OF EQUATION (63)

This Appendix proves that the function $\gamma(t)$ which maximizes the functional

$$I = \int_0^{\mathcal{T}} \left\{ -\lambda x^2(\sigma) + \int_0^{\mathcal{T}} x(\sigma)\, x(\rho) \left[R_h(\sigma-\rho) + \sum_{k=1}^{N} 2\beta_k R_h(\sigma-\rho-k\mathcal{T}) \right] d\rho \right\} d\sigma \qquad \text{(C-1)}$$

is a solution of the integral equation

$$\lambda\gamma(t) = \int_0^{\mathcal{T}} \gamma(\tau)\, K(t-\tau)\, d\tau \qquad 0 \leqslant t \leqslant \mathcal{T} \qquad \text{(C-2)}$$

where

$$K(t-\tau) \triangleq R_h(t-\tau) + \sum_{k=1}^{N} \beta_k \left[R_h(t-s+k\mathcal{T}) + R_h(t-s-k\mathcal{T})\right] \quad .$$

Following standard variational techniques,[64,85] let $x(t) = \gamma(t) + \epsilon\, f(t)$, where $\gamma(t)$ is the desired solution, and set

$$\left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} = 0 \quad .$$

From Eq. (C-1), this gives

$$\left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} = \int_0^{\mathcal{T}} \left\{ -2\lambda\gamma(\sigma)\, f(\sigma) + \int_0^{\mathcal{T}} [\gamma(\sigma)\, f(\rho) + f(\sigma)\, \gamma(\rho)] \right.$$

$$\left. \times \left[R_h(\sigma-\rho) + \sum_{k=1}^{N} 2\beta_k R_h(\sigma-\rho-k\mathcal{T}) \right] d\rho \right\} d\sigma \quad . \qquad \text{(C-3)}$$

However, since $R_h(\sigma-\rho) = R_h(\rho-\sigma)$, it follows that

$$\int_0^{\mathcal{T}}\int_0^{\mathcal{T}} \gamma(\sigma)\, f(\rho)\, R_h(\sigma-\rho)\, d\sigma\, d\rho = \int_0^{\mathcal{T}}\int_0^{\mathcal{T}} f(\sigma)\, \gamma(\rho)\, R_h(\sigma-\rho)\, d\sigma\, d\rho$$

and

$$\int_0^{\mathcal{T}}\int_0^{\mathcal{T}} \gamma(\sigma)\, f(\rho) \sum_{k=1}^{N} \beta_k R_h(\sigma-\rho-k\mathcal{T})\, d\sigma\, d\rho = \int_0^{\mathcal{T}}\int_0^{\mathcal{T}} \gamma(\rho)\, f(\sigma) \sum_{k=1}^{N}$$

$$\times\, \beta_k R_h(\sigma-\rho+k\mathcal{T})\, d\sigma\, d\rho \quad .$$

Thus, Eq. (C-3) can be written as

$$\left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} = 2\int_0^{\mathcal{T}} \left\{ -\lambda\gamma(\sigma) + \int_0^{\mathcal{T}} \gamma(\rho)\, K(\sigma-\rho)\, d\rho \right\} f(\sigma)\, d\sigma \qquad \text{(C-4)}$$

where $K(\sigma - \rho)$ is defined in Eq. (C-2). Finally, setting

$$\left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} = 0$$

for all f(t) leads to the result that the bracketed quantity in the integrand of Eq. (C-4) must be zero for all $0 \leqslant \sigma \leqslant \mathcal{T}$, i.e., $\gamma(t)$ must be a solution of Eq. (C-2).

APPENDIX D
DERIVATION OF EQUATION (70)

This Appendix proves that if $Q_\sigma(p)$ is a polynomial differential operator of order n, if v(t) has a continuous $(n-1)^{st}$ derivative and satisfies the boundary conditions of Eq. (68), and if K(t) is n times differentiable, then

$$\int_0^{\mathcal{T}} [Q_\sigma(p)\, v(\sigma)]\, K(t-\sigma)\, d\sigma = \int_0^{\mathcal{T}} v(\sigma)\, [Q_\sigma(-p)\, K(t-\sigma)]\, d\sigma \quad . \tag{D-1}$$

The definition of $Q_\sigma(p)$ implies that

$$Q_\sigma(p)\, v(\sigma) = \sum_{i=0}^{n} b_i v^{(i)}(\sigma) \qquad 0 \leqslant \sigma \leqslant \mathcal{T}$$

where

$$v^{(i)}(\sigma) \triangleq \frac{d^i v}{d\sigma^i}$$

and

$$Q(s) \triangleq \sum_{i=0}^{n} b_i s^i \quad .$$

Thus

$$\int_0^{\mathcal{T}} [Q_\sigma(p)\, v(\sigma)]\, K(t-\sigma)\, d\sigma = \sum_{i=0}^{n} b_i \int_0^{\mathcal{T}} v^{(i)}(\sigma)\, K(t-\sigma)\, d\sigma \quad .$$

For $i > 0$, integration by parts and substitution of the boundary conditions of Eq. (68) shows that

$$\int_0^{\mathcal{T}} v^{(i)}(\sigma)\, K(t-\sigma)\, d\sigma = K(t-\sigma)\, v^{(i-1)}(\sigma)\Big|_{\sigma=0}^{\sigma=\mathcal{T}} - \int_0^{\mathcal{T}} v^{(i-1)}(\sigma)\, \frac{d}{d\sigma} K(t-\sigma)\, d\sigma$$

$$= -\int_0^{\mathcal{T}} v^{(i-1)}(\sigma)\, \frac{d}{d\sigma} K(t-\sigma)\, d\sigma \quad .$$

Repeated application of this result leads to

$$\int_0^{\mathcal{T}} v^{(i)}(\sigma)\, K(t-\sigma)\, d\sigma = (-1)^i \int_0^{\mathcal{T}} v(\sigma)\, \frac{d^i}{d\sigma^i} K(t-\sigma)\, d\sigma \quad .$$

Thus,

$$\sum_{i=0}^{n} b_i \int_0^{\mathcal{T}} v^{(i)}(\sigma)\, K(t-\sigma)\, d\sigma = \int_0^{\mathcal{T}} v(\sigma) \left[\sum_{i=0}^{n} b_i (-1)^i \frac{d^i}{d\sigma^i} K(t-\sigma)\right] d\sigma$$

$$= \int_0^{\mathcal{T}} v(\sigma)\, [Q_\sigma(-p)\, K(t-\sigma]\, d\sigma$$

and Eq. (D-1) is proved.

APPENDIX E
PROOF OF EVEN-ODD PROPERTY OF NONDEGENERATE EIGENFUNCTIONS

It is desired to prove that all nondegenerate eigenfunctions of the eigenvalue differential equation

$$[N(p^2) - \lambda_i D(p^2)]\, v_i(t) = 0 \qquad -\frac{\mathcal{T}}{2} < t < \frac{\mathcal{T}}{2} \tag{E-1}$$

$$\begin{aligned} v_i(\pm \tfrac{\mathcal{T}}{2}) &= 0 \\ &\vdots \\ v_i^{(n-1)}(\pm \tfrac{\mathcal{T}}{2}) &= 0 \end{aligned} \tag{E-2}$$

are either even or odd functions. Observe first that since $N(p^2)$ and $D(p^2)$ contain only even order derivatives and since

$$\frac{d^{2k}}{dt^{2k}}\, v_i(t) = \frac{d^{2k}}{dt^{2k}}\, v_i(-t) \qquad k = 0, 1, \ldots \tag{E-3}$$

it follows that if $v_i(t)$ is an eigenfunction with eigenvalue λ_i, then $v_i(-t)$ is also an eigenfunction with eigenvalue λ_i. Thus, if $v_i(t)$ is a nondegenerate eigenfunction, there must exist a constant b such that

$$v_i(t) + b v_i(-t) = 0 \quad . \tag{E-4}$$

However, if $v_{ie}(t)$ and $v_{io}(t)$ denote the even and odd parts of $v_i(t)$, respectively, that is,

$$v_{ie}(t) = \frac{v_i(t) + v_i(-t)}{2}$$

and

$$v_{io}(t) = \frac{v_i(t) - v_i(-t)}{2}$$

it follows from Eq. (E-4) that $(1 - b)\, v_{ie}(t) = -(1 + b)\, v_{io}(t)$ and therefore that either $b = 1$ and $v_{io}(t) = 0$ or $b = -1$ and $v_{ie}(t) = 0$. In other words, a nondegenerate eigenfunction is either even or odd.

APPENDIX F
OPTIMUM TIME-LIMITED SIGNALS FOR COLORED NOISE

This Appendix presents a generalization of the results of Sec. IV-A to channels with colored noise. Since a derivation of the results for colored noise follows the previous work quite closely, only the final results are presented here.

The problem considered is that of choosing a fixed energy, time-limited (to $[0, \mathcal{T}]$) signal to give at the optimum detector[54] output both zero intersymbol interference at $t = k\mathcal{T}$, $k = \pm 1, \pm 2, \ldots$, and a maximum SNR at $t = 0$. Under the restrictions that $h(t)$ is a lumped parameter system and $N(f)$ is a rational spectrum, the following result is obtained. As before, let

$$H(s)\, H(-s) \triangleq \frac{N(s^2)}{D(s^2)}$$

let

$$N(f) \triangleq \left. \frac{\overline{N}(s^2)}{\overline{D}(s^2)} \right|_{s^2=-4\pi^2 f^2}$$

and assume that the orders of the polynomials $N(s^2)$, $D(s^2)$, $\overline{N}(s^2)$, and $\overline{D}(s^2)$ are $2m$, $2n$, $2\overline{m}$, $2\overline{n}$, respectively. Then the maximization problem previously outlined leads to the boundary value differential equation given by

$$[N(p^2)\, \overline{D}(p^2) - \lambda_i D(p^2)\, \overline{N}(p^2)]\, v_i(t) = 0 \qquad 0 < t < \mathcal{T}$$

$$v_i(0) = \qquad v_i(\mathcal{T}) = 0$$

$$\vdots$$

$$v_i^{(n+\overline{m}-1)}(0) = v_i^{(n+\overline{m}-1)}(\mathcal{T}) = 0 \quad . \tag{F-1}$$

The corresponding channel input signals $\{\gamma_i(t)\}$ are

$$\gamma_i(t) \triangleq \frac{1}{\sqrt{c}}\, Q(p)\, v_i(t) \tag{F-2}$$

where

$$Q(s) \triangleq \prod_{i=1}^{n+\overline{m}} (s - z_i)$$

$$z_i = \pm s_i \qquad\qquad i = 1, 2, \ldots, n + \overline{m}$$

$$s_i = \begin{cases} \text{RHP zeros of } D(s^2) & \text{for } i = 1, \ldots, n \\ \text{RHP zeros of } \overline{N}(s^2) & \text{for } i = n+1, \ldots, \overline{m} + n \end{cases}$$

and therefore

$$Q(s)\,Q(-s) = cD(s^2)\,\overline{N}(s^2) \quad .$$

These solutions have the following properties:

(1) The choice $z_i = s_i$ or $z_i = -s_i$ in the definition of Q(s) is arbitrary. Thus, there are $2^{n+\overline{m}}$ equally valid solutions to the maximization problem.

(2) The $\{\gamma_i(t)\}$ are orthogonal and may be assumed normalized, i.e., in the notation of Sec. IV-A,

$$(\gamma_i, \gamma_j)_{\mathcal{T}} = \delta_{ij} \quad .$$

This normalization is assumed hereafter.

(3) If $K_1(t)$ is defined as

$$K_1(t) \triangleq \int_{-\infty}^{\infty} \frac{1}{N(f)}\, e^{j\omega t}\, df$$

and if the generalized inner product is defined as

$$(f, K_1 g)_{\infty} \triangleq \int_{-\infty}^{\infty} f(t) \int_{-\infty}^{\infty} g(\sigma)\, K_1(t-\sigma)\, d\sigma\, dt$$

then the $\{\gamma_i(t)\}$ are "doubly orthogonal" (in the sense of Sec. IV-A) at the channel output with respect to the generalized inner product, i.e., if $r_{ij}(t)$ is the channel filter output when $\gamma_i(t - j\mathcal{T})$ is transmitted, then

$$(r_{ij}, K_1 r_{k\ell})_{\infty} = \begin{cases} \lambda_i & \text{if } i = k \text{ and } j = \ell \\ 0 & \text{otherwise} \end{cases} \quad .$$

(4) The eigenvalues $\{\lambda_i\}$ are the "generalized energy transfer ratios" of the filter for the corresponding eigenfunctions, i.e., with $r_{ij}(t)$ as previously defined,

$$\lambda_i = \frac{(r_{io}, K_1 r_{io})_{\infty}}{(\gamma_i, \gamma_i)_{\mathcal{T}}} \quad .$$

The importance of this result lies in the fact that the SNR at the optimum detector output at t = 0 is given by

$$\int_{-\infty}^{\infty} \frac{|X(f)\,H(f)|^2}{N(f)}\, df$$

when x(t) is the transmitted signal. Thus, $\gamma_1(t)$ is the solution to the original maximization problem.

ACKNOWLEDGMENT

I would like to express my sincere appreciation to Professor J.M. Wozencraft for his guidance and encouragement during this work. The opportunity of working closely with him for the past two years has proved to be the most stimulating and rewarding experience in my doctoral program. Also, I wish to thank my readers, Professor R.G. Gallager and Professor W.M. Siebert, for numerous helpful discussions, and to acknowledge the important role in this work of the bounding techniques developed by Professor Gallager.

I also wish to express my thanks to a number of people at Lincoln Laboratory who contributed significantly to this work. In particular, I would like to thank both P.E. Green, Jr., who provided the initial encouragement to work on this problem as well as the technical support that made the experimental program possible, and P.R. Drouilhet who provided valuable guidance both in the construction of the experimental equipment and in numerous discussions throughout the course of this work.

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DOCUMENT CONTROL DATA – DD 1473

1. ORIGINATING ACTIVITY Lincoln Laboratory, M.I.T.	2a. REPORT CLASSIFICATION Unclassified 2b. DOWNGRADING GROUP

3. REPORT TITLE

Digital Communication over Fixed Time-Continuous Channels with Memory – with Special Application to Telephone Channels

4. TYPE OF REPORT AND INCLUSIVE DATES

Technical Report

5. AUTHOR(S) *(Last name first)*

Holsinger, J.L.

6. REPORT DATE	7a. NO. OF PAGES	7b. NO. OF REFS.
20 October 1964	124	99

8a. CONTRACT NO. AF 19(628)-500	9a. ORIGINATOR'S REPORT NO. Technical Report 366
8b. ORDER NO. ARPA Order 512	9b. OTHER REPORT NO(S). ESD TDR-64-564; Research Laboratory of Electronics Technical Report 430

10. AVAILABILITY OR LIMITATION NOTICES

11. SUPPLEMENTARY NOTES	12. SPONSORING ACTIVITY Advanced Research Projects Agency

13. ABSTRACT

The objective of this study is to determine the performance, or a bound on the performance, of the "best possible" method for digital communication over fixed time-continuous channels with memory, i.e., channels with intersymbol interference and/or colored noise. The channel model assumed is a linear, time-invariant filter followed by additive, colored Gaussian noise. A general problem formulation is introduced which involves use of this channel once for T seconds to communicate one of M signals. Two questions are considered: (1) given a set of signals, what is the probability of error? and (2) how should these signals be selected to minimize the probability of error? It is shown that answers to these questions are possible when a suitable vector space representation is used, and the basis functions required for this representation are presented. Using this representation and the random coding technique, a bound on the probability of error for a random ensemble of signals is determined and the structure of the ensemble of signals yielding a minimum error bound is derived. The inter-relation of coding and modulation in this analysis is discussed and it is concluded that: (1) the optimum ensemble of signals involves an impractical modulation technique, and (2) the error bound for the optimum ensemble of signals provides a "best possible" result against which more practical modulation techniques may be compared. Subsequently, several suboptimum modulation techniques are considered, and one is selected as practical for telephone channels. A theoretical analysis indicates that this modulation system should achieve a data rate of about 13,000 bits/second on a data grade telephone line with an error probability of approximately 10^{-5}. An experimental program substantiates that this potential improvement could be realized in practice.

14. KEY WORDS

time-continuous channel
memory devices
telephone communications

Printed by
United States Air Force
L. G. Hanscom Field
Bedford, Massachusetts